Deriving Corrective Permeability from the Cumulative Deviation Functional; Robert Galida (June 2026) [F]

Abstract

The attractor framework defines κ (corrective permeability) as the rate at which a system returns to its attractor after perturbation. Historically, κ has been treated as an empirical parameter — fitted to data rather than derived from first principles. This paper derives κ from the framework's foundational object: the cumulative deviation functional $D_T(x)=\int_0^T \delta(\phi_t(x))\,dt$, where $\delta(x)=d(x,A)$.

We define: $\kappa=\inf_{x\in B\setminus A}\frac{\delta(x)}{D_\infty(x)}$

We prove that for linear systems $\dot{x}=-Ax$ with A symmetric positive definite, this definition recovers the slowest eigenvalue $\lambda_{\min}(A)$ — the conventional notion of corrective permeability. We establish a sharp universal persistence bound $D_\infty(x)\le\delta(x)/\kappa$, show homogeneity and scale invariance of the variational ratio, and demonstrate consistency with Koopman spectral theory and resolvent poles for finite-dimensional linear systems. A comparison theorem links κ to classical exponential stability constants. A Hamilton-Jacobi-type transport equation for D_∞ is derived. A finite-horizon estimator $\kappa_T=\inf_x\frac{\delta(x)}{D_T(x)}$ is provided with exponential convergence under explicit assumptions.

The derivation is rigorous for linear systems and testable. Open questions for nonlinear, multiscale, and stochastic systems are identified.

Keywords: corrective permeability, cumulative deviation functional, attractor framework, Koopman operator, trajectory functional

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1. Introduction

The attractor framework has been applied across physics, biology, cognition, and social systems. Its central variable — corrective permeability κ — measures the rate at which a system returns to its attractor after perturbation. Historically, κ has been defined empirically as $\kappa = 1/\tau$, where τ is a measured recovery time constant.

This paper derives κ from a single foundational object: the cumulative deviation functional $D_T(x)$. Within the present framework, κ is defined variationally rather than introduced as an empirical fitting parameter. We show that κ is a consequence of the trajectory geometry — specifically, the ratio of initial distance to total cumulative deviation.

The derivation is rigorous for linear systems, connects to established theory (Koopman operators, resolvent poles), and provides a finite-horizon estimator for empirical use. Open questions for nonlinear and stochastic systems are identified.

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2. The Cumulative Deviation Functional

Let X be a metric space with distance function $\|\cdot - \cdot\|$. Let $\phi_t(x)$ be the flow of a dynamical system starting from state $x \in X$ at time $t=0$. Let $A \subseteq X$ be an attractor set (a compact, invariant set to which trajectories converge). Let B be the basin of attraction of A.

Define the distance from a point to the attractor:$$\delta(x) = d(x, A) = \inf_{a \in A} \|x - a\|$$

Definition 1 (Cumulative Deviation Functional): For a finite horizon $T>0$, define:$$D_T(x) = \int_0^T \delta(\phi_t(x))\, dt$$

For $T \to \infty$, define:$$D_\infty(x) = \int_0^\infty \delta(\phi_t(x))\, dt$$

Proposition 1 (Finiteness of D_∞): Assume there exist constants $C<\infty$ and $\mu>0$ such that:$$\delta(\phi_t(x)) \le C e^{-\mu t} \delta(x)$$

for all $x \in B$. Then $D_\infty(x) < \infty$ for every $x \in B$.

Proof:$$D_\infty(x) = \int_0^\infty \delta(\phi_t(x))\, dt \le \int_0^\infty C e^{-\mu t} \delta(x)\, dt = \frac{C}{\mu} \delta(x) < \infty \quad \square$$

Properties (from Galida, 2026a):

Property	Statement
Non-negativity	$D_T(x) \ge 0$
Monotonicity	$D_{T_2}(x) \ge D_{T_1}(x)$ for $T_2 \ge T_1$
Additivity	$D_{T+S}(x) = D_T(x) + D_S(\phi_T(x))$
Instantaneous growth	$\frac{d}{dT} D_T(x) = \delta(\phi_T(x))$

Property	Statement
Occupation measure	$D_T(x)=\int \delta(y)\, d\mu_T(y)$, where μ_T is the occupation measure

3. Derivation of Corrective Permeability (κ)

3.1 Variational Definition

Definition 2 (Corrective Permeability): $\kappa = \inf_{x \in B \setminus A} \frac{\delta(x)}{D_\infty(x)}$

Interpretation: κ is the *effective* recovery rate — the smallest ratio of initial distance to total cumulative deviation. It serves as a global measure of the slowest recovery mode in the basin.

Remark on κ: The definition allows $\kappa=0$ if $D_\infty(x)$ diverges or if the ratio $\delta(x)/D_\infty(x)$ can be made arbitrarily small. Throughout the remainder of this paper, we assume hypotheses (such as the exponential stability in Proposition 1) that guarantee $\kappa>0$.

Remark on attainment: The infimum in the definition of κ need not be attained; minimizing sequences may exist without a minimizing state. For linear systems, the infimum is attained on the slow eigenspace.

3.2 Homogeneity and Scale Invariance

Theorem 1 (Homogeneity and Scale Invariance): Suppose the flow satisfies $\phi_t(\alpha x)=\alpha\phi_t(x)$ for all t and all $\alpha>0$, and the distance function satisfies $\delta(\alpha x)=\alpha\delta(x)$. Then: $$\frac{D_\infty(\alpha x)}{\delta(\alpha x)}=\frac{D_\infty(x)}{\delta(x)}$$

Proof: $$D_\infty(\alpha x)=\int_0^\infty \delta(\phi_t(\alpha x))\,dt=\int_0^\infty \delta(\alpha\phi_t(x))\,dt=\alpha\int_0^\infty \delta(\phi_t(x))\,dt=\alpha D_\infty(x)$$

Corollary: For linear systems, the infimum over all $x\neq 0$ reduces to an infimum over the unit sphere: $$\kappa=\inf_{\|x\|=1}\frac{\delta(x)}{D_\infty(x)}$$

3.3 Sharp Universal Persistence Bound

Theorem 2 (Sharp Universal Persistence Bound): For any $x\in\mathcal{B}_A$: $$D_\infty(x)\le\frac{\delta(x)}{\kappa}$$

Moreover, the constant $1/\kappa$ is optimal: it is the *smallest* constant such that this inequality holds for all x in the basin.

Proof: By definition of κ as the infimum of $\delta(x)/D_\infty(x)$, we have $\delta(x)/D_\infty(x)\ge\kappa$ for all x. Rearranging gives: $$D_\infty(x)\le\frac{\delta(x)}{\kappa}$$

Optimality follows from Theorem 3: for the slow eigenvector v_1, $D_\infty(v_1)=\delta(v_1)/\kappa$, so no smaller constant can work. □

3.4 Consistency with Linear Systems

Consider a linear system $\dot{x}=-Ax$, with A symmetric positive definite. Let its eigenvalues be $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n$, with corresponding orthonormal eigenvectors $v_1,v_2,\ldots,v_n$.

The flow is $\phi_t(x) = e^{-At}x$. The attractor is $\mathcal{A}=\{0\}$, and the distance to the attractor is $\delta(x)=\|x\|$.

Theorem 3 (Linear Consistency): For $\dot{x}=-Ax$ with A symmetric positive definite,
$$\inf_{x\neq 0}\frac{\|x\|}{D_\infty(x)} = \lambda_{\min}(A)$$

Proof:

Since A is symmetric positive definite, e^{-At} is symmetric positive definite with eigenvalues $e^{-\lambda_i t}$. Hence its operator norm is $\|e^{-At}\| = e^{-\lambda_1 t}$. For any $x\neq 0$:
$$D_\infty(x)=\int_0^\infty \|e^{-At}x\|\,dt \le \int_0^\infty \|x\| e^{-\lambda_1 t}\,dt = \frac{\|x\|}{\lambda_1}$$

Therefore:
$$\frac{\|x\|}{D_\infty(x)} \ge \lambda_1$$

To show equality is achieved, take $x=v_1$ (the eigenvector corresponding to λ_1). Then:
$$\|e^{-At}v_1\| = \|v_1\| e^{-\lambda_1 t}$$

and:
$$D_\infty(v_1)=\int_0^\infty \|v_1\| e^{-\lambda_1 t}\,dt = \frac{\|v_1\|}{\lambda_1}$$

Thus:
$$\frac{\|v_1\|}{D_\infty(v_1)} = \lambda_1$$

Hence:
$$\inf_{x\neq 0}\frac{\|x\|}{D_\infty(x)} = \lambda_1 \quad \square$$

Corollary: For linear systems, the variational definition of κ recovers the slowest eigenvalue — the conventional notion of corrective permeability.

3.5 Transport Equation

Theorem 4 (Transport Equation): Assume the vector field f is C^1, the flow ϕ_t is C^1, and D_∞ is continuously differentiable on $\mathcal{B}\setminus\mathcal{A}$.

Then: $\nabla D_\infty(x) \cdot f(x) = -\delta(x)$

Proof: From the definition: $D_\infty(\phi_s(x)) = D_\infty(x) - D_s(x)$

Differentiating with respect to s at $s=0$: $\frac{d}{ds} D_\infty(\phi_s(x))\big|_{s=0} = -\delta(x)$

By the chain rule: $\nabla D_\infty(x) \cdot f(x) = -\delta(x)$ $\square$

Interpretation: This is a first-order transport equation, $f \cdot \nabla D = -\delta$, which belongs to the broader Hamilton-Jacobi family but lacks a Hamiltonian in the usual sense. It may serve as a foundation for numerical computation and further theoretical development.

3.6 Local vs. Global Interpretation

The variational definition $\kappa = \inf_x \frac{\delta(x)}{D_\infty(x)}$ is **global** — it is the slowest recovery rate over the entire basin. This is not necessarily the same as the local recovery rate near the attractor (the slowest eigenvalue of the linearization). For linear systems, they coincide. For nonlinear systems, they may differ if transient excursions produce slower effective recovery than the local linearization predicts.

This distinction is important: κ is a global invariant of the basin, not merely a local property of the attractor. The relationship between the global κ and the local Lyapunov exponent is an open question (see §6).

3.7 Non-Symmetric Linear Systems

For a general linear system $\dot{x} = Ax$ (where A is stable, i.e., all eigenvalues have negative real parts), the same principle holds in the diagonalizable case. The slowest mode corresponds to the eigenvalue with the largest real part (closest to zero).

Conjecture: An analogous result holds for non-normal linear systems under additional assumptions on the semigroup, such as a uniformly exponentially stable semigroup satisfying suitable norm bounds. This remains an open question.

3.8 Comparison with Exponential Stability

Theorem 5 (Comparison with Exponential Stability): Suppose the system satisfies the exponential stability bound:

$$\delta(\phi_t(x)) \le C e^{-\mu t} \delta(x)$$

for all $x \in B$, with constants $C < \infty$ and $\mu > 0$. Then:

$$\kappa \ge \frac{\mu}{C}$$

Proof: From the stability bound:

$$D_\infty(x) = \int_0^\infty \delta(\phi_t(x))\, dt \le \int_0^\infty C e^{-\mu t} \delta(x)\, dt = \frac{C}{\mu} \delta(x)$$

Therefore:

$$\frac{\delta(x)}{D_\infty(x)} \ge \frac{\mu}{C}$$

Taking the infimum over x:

$$\kappa = \inf_x \frac{\delta(x)}{D_\infty(x)} \ge \frac{\mu}{C} \quad \square$$

Interpretation: The variational constant κ is bounded below by the exponential stability constant μ/C.

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4. Connections to Existing Theory

4.1 Koopman Operator

The Koopman operator K^t acts on observables as:

$$(K^t f)(x) = f(\phi^t(x))$$

For linear systems $\dot{x} = -Ax$, the Koopman eigenvalues are $e^{-\lambda_i t}$. The dominant nontrivial eigenvalue (largest less than 1) is $e^{-\lambda_1 t}$, corresponding to the slowest decay rate.

For finite-dimensional linear systems, $\rho = e^{-\lambda_{\min} t}$, and therefore:

$$-\frac{1}{t}\log \rho = \lambda_{\min} = \kappa$$

Thus, under the hypotheses of Theorem 3, the variational constant equals the exponential decay rate associated with the dominant Koopman eigenvalue.

4.2 Resolvent Poles

For finite-dimensional stable linear systems, the resolvent $(sI + A)^{-1}$ has poles at $s = -\lambda_i$. The pole closest to the imaginary axis is $s = -\lambda_1$.

Since Theorem 3 identifies $\kappa = \lambda_{\min}$, and the resolvent poles are $s_i = -\lambda_i$, we obtain:

$$\kappa = \min_i |\Re(s_i)|$$

for finite-dimensional linear systems.

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5. Finite-Horizon Estimation

In practice, we can only measure finite trajectories. Define the finite-horizon estimator:

$$\kappa_T = \inf_{x \in K} \frac{D_T(x)}{\delta(x)}$$

where $K \subset B$ is compact and $K \cap A = \varnothing$.

Proposition 2 (Finite-Horizon Estimation): Assume:

1. The flow $\phi_t(x)$ is jointly continuous in (t,x).
2. $\delta(x)$ is continuous.
3. The exponential stability bound $\delta(\phi_t(x)) \le C e^{-\mu t} \delta(x)$ holds uniformly for all $x \in K$, with $\mu > 0$.

Then the variational constant κ (from Definition 2) satisfies $\kappa \ge \mu / C$ by Theorem 5, and:

$$\kappa_T \to \kappa \quad \text{as } T \to \infty$$

with error:

$$|\kappa_T - \kappa| = O(e^{-\mu T})$$

Proof: For any $x \in K$, the tail bound gives:

$$|D_\infty(x) - D_T(x)| = \int_T^\infty \delta(\phi_t(x))\, dt \le \frac{C e^{-\mu T} \delta(x)}{\mu}$$

Since $\delta(x)$ is bounded on the compact set K, let $M = \sup_{x \in K} \delta(x) < \infty$. Then:

$$|D_\infty(x) - D_T(x)| \le \frac{C M e^{-\mu T}}{\mu}$$

The right-hand side is independent of x and tends to zero as $T \to \infty$. Hence $D_T \to D_\infty$ uniformly on K.

Moreover, since K is compact and $K \cap A = \varnothing$, continuity of δ gives $\inf_{x \in K} \delta(x) > 0$. Since $D_T(x)$ is continuous (by assumptions 1–2) and monotonically non-decreasing in T (from §2), for any fixed finite $T_0 > 0$, $D_\infty(x) \ge D_{T_0}(x)$, and D_{T_0} is continuous and strictly positive on K. A continuous, strictly positive function on a compact set has a positive infimum: $m = \inf_{x \in K} D_{T_0}(x) > 0$

Thus: $\inf_{x \in K} D_\infty(x) \geq m > 0$

Uniform convergence of D_T to D_∞ on K therefore implies uniform convergence of $\delta(x)/D_T(x)$ to $\delta(x)/D_\infty(x)$. Consequently, the infima converge. □

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6. Open Questions

Question	Status	Difficulty
Q1: Nonlinear systems	Does $\inf \frac{\delta}{D_\infty}$ equal the local Lyapunov exponent?	Hard
Q2: Local vs. global consistency	Does $\lim_{x \to A} \frac{\delta(x)}{D_\infty(x)} = \kappa$ hold for general nonlinear systems?	Hard
Q3: Non-normal systems	Does the infimum equal the slowest eigenvalue for non-normal A?	Moderate
Q4: Multiple timescales	Does the infimum isolate the slowest timescale?	Hard
Q5: Stochastic systems	How does noise affect the finite-horizon estimator?	Hard
Q6: Multiple attractors	How does κ behave in basins with multiple attractors?	Moderate

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7. Conclusion

This paper derives corrective permeability κ from the cumulative deviation functional $D_T(x)$. The variational definition:

$$\kappa = \inf_x \frac{\delta(x)}{D_\infty(x)}$$

is shown to recover the slowest eigenvalue for linear systems, consistent with the conventional empirical definition $\kappa = 1/\tau$. A sharp universal persistence bound $D_\infty(x) \le \delta(x)/\kappa$ is established. A comparison theorem links κ to classical exponential stability constants. A Hamilton-Jacobi-type transport equation for D_∞ is derived. Connections to Koopman theory and resolvent theory are established for finite-dimensional linear systems. A finite-horizon estimator κ_T is provided with exponential convergence under explicit assumptions.

Key contribution: Within the present framework, κ is defined variationally rather than introduced as an empirical fitting parameter — at least for the class of systems analyzed here.

Next steps: Extend the derivation to nonlinear systems (Q1–Q2), non-normal systems (Q3), multiple timescales (Q4), and stochastic dynamics (Q5).

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