

Excess Entropy Production as a Candidate Universal Cost of Persistence: A Thermodynamic Foundation for the Attractor Framework; Robert Galida (July 2026) [F]

Abstract

Every dissipative system maintains its attractor through continuous reconfiguration. Reconfiguration requires work; work generates entropy. The recovery rate κ – corrective permeability – is the rate at which a system reconfigures to return to its attractor after perturbation. This paper proposes that κ is a measure of excess entropy generation rate.

We develop an abstract persistence cost framework and prove its equivalence to Lyapunov theory. We then identify entropy production as a physical realization of this cost, deriving: $\kappa = \inf_x \delta(x) \int_0^\infty \sigma_{\text{excess}}(\phi^t(x)) dt$ $\kappa = x \inf_x \int_0^\infty \sigma_{\text{excess}}(\phi^t(x)) dt \delta(x)$

where $\sigma_{\text{excess}} = \sigma - \sigma_{ss}$ $\sigma_{\text{excess}} = \sigma - \sigma_{ss}$ is the excess entropy production rate above the system's steady-state baseline. For physical systems, the baseline is zero (equilibrium); for biological, cognitive, and social systems, the baseline is the steady-state dissipation rate of the healthy, well-coordinated attractor.

This unifies physical, biological, cognitive, and social systems. The framework is grounded in the second law of

thermodynamics and non-equilibrium steady-state thermodynamics, not analogy. Empirical predictions are provided for each domain.

Keywords: entropy generation, excess entropy production, corrective permeability, attractor framework, dissipative structures, reconfiguration, Lyapunov theory, free energy principle, allostatic load

1. Introduction

The attractor framework defines persistence as the ability of a system to maintain its attractor under perturbation. Historically, persistence has been measured kinematically – as distance traveled or time spent away from equilibrium. This paper proposes that the true cost of persistence is thermodynamic: it is the excess entropy generated during reconfiguration and recovery.

Every dissipative system maintains its attractor through continuous reconfiguration. A bacterium reconfigures its metabolism to maintain homeostasis. A brain reconfigures its synaptic connections to maintain predictive models. A society reconfigures its institutions to maintain order. Reconfiguration requires work; work generates entropy. The second law of thermodynamics applies at every level of organization.

We develop an abstract persistence cost framework first, establishing its equivalence to Lyapunov theory. We then identify entropy production as a physical realization of this cost, deriving the relationship between corrective permeability and excess entropy generation.

The framework unifies physical, biological, cognitive, and

social systems. It is grounded in the second law of thermodynamics and non-equilibrium steady-state thermodynamics, not analogy.

2. The Persistence Cost Functional

Let X be a state space, $\phi_t(x)$ the flow of a dynamical system, and $A \subseteq X$ an attractor set. Let $\delta(x) = d(x, A)$ be the distance from x to the attractor. For a treatment of state-space constraints in viability theory, see Aubin (1991).

Definition 1 (Persistence Cost Functional): A persistence cost functional $C(x)$ is a scalar function on X satisfying:

1. $C(x) \geq 0$ for all x
2. $C(x) = 0$ if and only if $x \in A$
3. $C(\phi_t(x)) \in L^1([0, \infty))$ for all x in the basin

Definition 2 (Cumulative Persistence Cost): For a finite horizon $T > 0$: $D_T(x) = \int_0^T C(\phi_t(x)) dt$

For trajectories that converge to the attractor: $D_\infty(x) = \int_0^\infty C(\phi_t(x)) dt$

3. Existence and Lyapunov Equivalence

Theorem 1 (Existence of the Persistence Functional): Assume $C(x) \geq 0$, $C = 0$ only on A ,

and $C(\phi_t(x)) \in L^1([0, \infty))$ $C(\phi_{t^*}(x)) \in L^1([0, \infty))$ for all x in the basin. Assume f is locally Lipschitz, the flow is continuously differentiable in the initial condition, and C is continuous and locally bounded. Then:

1. $D^\infty(x) = \int_0^\infty C(\phi_t(x)) dt$ $D^\infty(x^*) = \int_0^\infty C(\phi_{t^*}(x)) dt$ exists and is finite.
2. $D^\infty D^\infty(x^*)$ is continuous.
3. $D^\infty D^\infty(x^*)$ satisfies the transport equation:

$$\nabla D^\infty(x) \cdot f(x) = -C(x) \quad \nabla D^\infty(x^*) \cdot f(x) = -C(x)$$

Proof: The integral exists and is finite by the $L^1 L^1$ assumption. Continuity follows from the dominated convergence theorem under the stated regularity assumptions. To derive the transport equation, compute: $D(\phi_h(x)) = \int_0^\infty C(\phi_t(x)) dt = D(x) - \int_0^h C(\phi_t(x)) dt$ $D(\phi_{h^*}(x)) = \int_0^\infty C(\phi_{t^*}(x)) dt = D(x) - \int_0^h C(\phi_{t^*}(x)) dt$

$$\text{Then: } D(\phi_h(x)) - D(x) = -\int_0^h C(\phi_t(x)) dt \rightarrow -C(x)h \quad D(\phi_{h^*}(x)) - D(x) = -\int_0^h C(\phi_{t^*}(x)) dt \rightarrow -C(x)$$

as $h \rightarrow 0$ $h \rightarrow 0$. By the chain rule: $\nabla D(x) \cdot f(x) = -C(x)$ $\nabla D(x^*) \cdot f(x) = -C(x)$ \square

Corollary (Equivalence to Lyapunov Theory): Any Lyapunov function $V(x)$ (with $V \geq 0$, $V = 0$ on the attractor, and $V' \leq 0$) yields a persistence cost $C(x) = -V'(x)$. Conversely, any persistence cost $C(x)$ satisfying $\nabla D \cdot f = -C$ defines a Lyapunov function $D(x)$.

Proof: If V is a Lyapunov function, then $V' = \nabla V \cdot f \leq 0$. Define $C = -V'$. Then $C \geq 0$, $C = 0$ on the attractor, and $D = \int C = V(x) - V(\phi_T(x))$. Conversely, if $\nabla D \cdot f = -C$, then $D' = -C \leq 0$, so D is a Lyapunov function. \square

Interpretation: The persistence cost framework is mathematically equivalent to classical Lyapunov stability theory. For the connection to contraction analysis, see Lohmiller & Slotine (1998). For control Lyapunov functions, see Freeman & Kokotovic (1996). Entropy production is one physically meaningful realization of the cost function CC . For a detailed treatment of Lipschitz continuity of $D^\infty D^\infty$ under a Lipschitz-flow hypothesis, see Galida (2026a), Proposition 4.

4. Entropy Production as Persistence Cost

4.1 Entropy Balance

For an open system, the entropy balance equation is: $dS_{\text{system}}/dt = \sigma - \Phi$

where $\sigma \geq 0$ is the entropy production rate (always non-negative by the second law) and Φ is the entropy export rate to the environment. For foundational treatments of stochastic thermodynamics and entropy production, see Seifert (2012) and Sekimoto (2010).

For a system in a steady state: $dS_{\text{system}}/dt = 0 \implies \sigma = \Phi$

4.2 Excess Entropy Production

Define the **steady-state entropy production rate** σ_{ss} as the rate when the system is at its attractor.

Define the **excess entropy production rate**: $\sigma_{\text{excess}}(x) = \sigma(x) - \sigma_{\text{ss}}$

Assumption (Excess Entropy Decay): For all trajectories in the

basin, there exist constants $C < \infty$ and $\mu > 0$ such that:

$$\sigma_{\text{excess}}(\phi^t(x)) \leq C e^{-\mu t} \sigma_{\text{excess}}(x)$$

$$\sigma_{\text{excess}}(\phi^t(x)) \leq C e^{-\mu t} \sigma_{\text{excess}}(x)$$

for all $t \geq 0$. This ensures $D^\infty(x) < \infty$ and is the standard hypothesis under which the persistence functional and its associated bounds are well-defined, consistent with Galida (2026a, 2026b). The decay rate μ may be domain-specific and is empirically measurable.

Note on generalization: The exponential decay assumption is adopted here to ensure finiteness of D^∞ and to maintain consistency with the prior papers in this series. Generalization to L1 integrable decays (e.g., algebraic) is a priority for future work.

4.3 The Entropy Persistence Functional

Definition 3 (Cumulative Excess Entropy Functional): For a finite horizon $T > 0$:

$$D^T(x) = \int_0^T \sigma_{\text{excess}}(\phi^t(x)) dt$$

$$D^T(x) = \int_0^T \sigma_{\text{excess}}(\phi^t(x)) dt$$

For trajectories that converge to the attractor:

$$D^\infty(x) = \int_0^\infty \sigma_{\text{excess}}(\phi^t(x)) dt$$

$$D^\infty(x) = \int_0^\infty \sigma_{\text{excess}}(\phi^t(x)) dt$$

Interpretation: The persistence functional is the total excess entropy generated during reconfiguration and recovery.

4.4 Corrective Permeability

Definition 4 (Corrective Permeability): $\kappa = \inf_{x \in B} \delta(x) D^\infty(x)$

$$\kappa = \inf_{x \in B} \delta(x) D^\infty(x)$$

where $\delta(x) = d(x, A)$ is the distance to the attractor.

Interpretation: κ is the minimum excess entropy cost per unit distance. It measures the efficiency of reconfiguration: a system that returns with minimal excess entropy generation has

high κ ; a system that generates excess entropy has low κ .

4.5 Basin Depth

Proposition 1 (Properties of Basin Depth): Define $B = D^\infty(\text{saddle}) - D^\infty(A)$, where saddle is the lowest point on the basin boundary (the separatrix between attractors). For the connection to large-deviation theory and escape rates, see Freidlin & Wentzell (2012). Then:

1. $B \geq 0$, with equality iff the basin has no barrier (i.e., the boundary coincides with the attractor).
2. For gradient systems $\dot{x} = -\nabla V(x)$, $B = V(\text{saddle}) - V(A)$ (the classical energy barrier).
3. B is invariant under smooth coordinate changes (coordinate invariance).
4. B depends on the chosen persistence cost functional C ; different costs yield different barriers.

Proof: (1) follows from non-negativity of D^∞ . (2) follows from the transport equation $\nabla \cdot f = -C$ and the identity $f = -\nabla V$. (3) follows from the invariance of the integral under diffeomorphisms. (4) is self-evident.

5. Domain-Specific Realizations

5.1 Physical Systems: Thermodynamic

Excess Entropy

For a thermodynamic system, $S(x) = k_B \log \Omega(x)$, where $\Omega(x)$ is the number of microstates. For an isolated system, $\sigma_{ss} = 0$ (equilibrium), so $\sigma_{\text{excess}} = \sigma - S'$. $\kappa = \inf_x \delta(x) [S(A) - S(x)]$

Example: A gas returning to equilibrium after compression. The entropy generated is $\Delta S = nR \log(V_f/V_i)$.

5.2 Biological Systems: Metabolic Excess Entropy

For a biological system, $S(x)$ is the metabolic entropy. The baseline σ_{ss} is the resting metabolic rate (homeostasis). The excess is: $\sigma_{\text{excess}} = \text{metabolic rate} - \text{resting metabolic rate}$. $\kappa = \inf_x \delta(x) \int_0^\infty \sigma_{\text{excess}}(\phi_t(x)) dt$

Example: A cell returning to homeostasis after a nutrient shock. The excess entropy generated is the metabolic cost of restoring homeostasis above baseline. For the dissipative-structures framework underlying biological self-organization, see Nicolis & Prigogine (1989).

5.3 Cognitive Systems: Free Energy Dissipation

For a cognitive system, variational free energy $F = -\log p(y|x) + \text{DKL}[q(\cdot) \| p(\cdot|x)]$ is adopted here as one candidate persistence functional. We do not claim variational free energy is uniquely correct; it is adopted as the most developed existing candidate persistence functional for cognitive systems. Other candidates (Bayesian surprise, expected free energy, predictive information) are possible; this paper focuses

on FF due to its established role in the free-energy principle (Friston, 2010). For the thermodynamics of information and its connection to free-energy minimization, see Parrondo, Horowitz & Sagawa (2015) and Sagawa & Ueda (2008).

The baseline σ_{ss} is the baseline neural dissipation rate (resting brain activity). The excess is: $\sigma_{\text{excess}} = F' - F'_{ss}$
 $\kappa = \int_0^\infty \int \sigma_{\text{excess}}(\phi_t(x)) dt \delta(x)$

Example: A cognitive system updating its beliefs after a prediction error. The excess entropy generated is the free energy dissipated during belief updating above baseline.

5.4 Social Systems: Coordination Excess Entropy

For a social system, define the aggregate social entropy production rate as: $\sigma_{\text{social}}(t) = \sum_i (S'_i(t) - S'_{i\text{rest}})$

where $S'_i(t)$ is the total entropy production rate of individual i , and $S'_{i\text{rest}}$ is the individual's baseline entropy production rate in a resting, minimally socially constrained state. This is measured via physiological proxies such as basal metabolic rate, resting allostatic load, or cortisol baseline (McEwen, 1998; Sterling & Eyer, 1988).

Interpretation: σ_{social} measures the excess dissipation attributable to social constraints: the additional entropy generated by coordination, communication, conflict, norm enforcement, and institutional friction.

Non-Negativity: Unlike total entropy production $S'_i \geq 0$ (which follows from the second law), σ_{social} is not guaranteed to be non-negative. Division of labor, infrastructure, and specialization may *reduce* an individual's

metabolic burden relative to a solitary baseline. The hypothesis is that during recovery from social disruption, $\sigma_{\text{social}} \geq 0$ $\sigma_{\text{social}} \rightarrow 0$; in steady-state, $\sigma_{\text{social}} \rightarrow 0$ $\sigma_{\text{social}} \rightarrow 0$. This is an empirical claim, not a theorem.

The baseline σ_{ss} is the steady-state social entropy production rate (well-coordinated society). The excess is: $\sigma_{\text{excess}} = \sigma_{\text{social}} - \sigma_{\text{ss}}$

$$\kappa = \int_0^\infty \int_{\mathcal{X}} \sigma_{\text{excess}}(\phi_t(x)) dt \delta(x) \quad \kappa = \int_0^\infty \int_{\mathcal{X}} \sigma_{\text{excess}}(\phi_t(x)) dt \delta(x)$$

Example: A society recovering from a shock (economic crisis, political upheaval). The excess entropy generated is the coordination cost of restructuring above baseline. A harmonious society has $\sigma_{\text{excess}} = 0$; a turbulent society has $\sigma_{\text{excess}} > 0$; a chronically turbulent society may have settled into a new attractor with a higher σ_{ss} . This illustrates the framework's central distinction: the attractor is the state of minimum entropy generation for that class of system.

6. The Unified Framework

6.1 Summary Table

Domain	Entropy Functional	Baseline σ_{ss}	Excess σ_{excess}	Recovery Rate κ
Physical	Thermodynamic entropy	0 (equilibrium)	$S - S'$	$\int_0^\infty \delta \Delta S dt$
Biological	Metabolic entropy	Resting metabolic rate	Metabolic rate - resting	$\int_0^\infty \delta \sigma_{\text{excess}} dt$
Cognitive	Free energy	Baseline neural dissipation	$F' - F'_{\text{ss}}$	$\int_0^\infty \delta \sigma_{\text{excess}} dt$
Social	Social entropy production	Steady-state social dissipation	$\sigma_{\text{social}} - \sigma_{\text{ss}}$	$\int_0^\infty \delta \sigma_{\text{excess}} dt$

6.2 The Universal Structure

Every domain follows the same mathematical structure:

Component	Expression
Excess entropy production	$\sigma_{\text{excess}}(x) = \sigma(x) - \sigma_{\text{ss}}$ $\sigma_{\text{excess}}[\cdot](x) = \sigma(x) - \sigma_{\text{ss}}[\cdot]$
Cumulative cost	$D^\infty(x) = \int_0^\infty \sigma_{\text{excess}}(\phi^t(x)) dt$ $D^\infty[\cdot](x) = \int_0^\infty \sigma_{\text{excess}}[\cdot](\phi^t(x)) dt$
Recovery rate	$\kappa = \inf_x \delta(x) / D^\infty(x)$ $\kappa = \inf_x \delta(x) / D^\infty[\cdot](x)$
Basin depth	$B = D^\infty(\text{saddle})$ $B = D^\infty[\cdot](\text{saddle})$
Transport equation	$\nabla D \cdot f = -\sigma_{\text{excess}}$ $\nabla D \cdot f = -\sigma_{\text{excess}}[\cdot]$

6.3 The Low-Energy Attractor Benchmark (Proposed Hypothesis)

We propose the following benchmark as an additional hypothesis: **the attractor is the state of minimum entropy generation for that class of system.**

Domain	Attractor	Entropy Generation at Attractor
Physical	Equilibrium	$\sigma = 0$ $\sigma = 0$
Biological	Homeostasis	$\sigma = \sigma_{\text{ss}} > 0$ $\sigma = \sigma_{\text{ss}}[\cdot] > 0$ (resting metabolism)
Cognitive	Settled Belief	$\sigma = \sigma_{\text{ss}} > 0$ $\sigma = \sigma_{\text{ss}}[\cdot] > 0$ (baseline neural dissipation)
Social	Coordinated Order	$\sigma = \sigma_{\text{ss}} > 0$ $\sigma = \sigma_{\text{ss}}[\cdot] > 0$ (baseline institutional friction)

Interpretation:

1. For **equilibrium systems** (gases, isolated systems), the attractor is the state where entropy generation reaches zero – the system has nowhere lower to go.

2. For **dissipative systems** (cells, brains, societies), the attractor is the state where entropy generation reaches its lowest *non-zero* steady-state value – the minimum entropy generation the system can sustain while maintaining its functional organization.

Important caveats:

- This is a proposed benchmark, not a derived theorem.
- For cognitive systems in particular, minimizing *entropy production rate* (a thermodynamic quantity) and minimizing *free energy/surprise* (the actual claim in the free-energy principle) are distinct minimization principles. The framework does not establish a bridge between them; this is an open question.
- The benchmark is an empirical hypothesis that requires domain-specific validation.

In all cases, the attractor is the **lowest entropy-generating state that system can have while remaining itself.**

7. Testable Predictions

7.1 Core Prediction

Prediction: The recovery rate κ is inversely proportional to the excess entropy generated during reconfiguration: $\kappa \propto 1/D_{\infty}$ $\kappa \propto D_{\infty}^{-1}$

Falsification: If a system returns to its attractor with high excess entropy generation but high recovery rate, the prediction is falsified.

7.2 Secondary Prediction

Prediction: Systems that maintain their attractor with minimal excess entropy generation are more “efficient.” Systems that generate excess entropy are “inefficient” or “stressed.”

Falsification: If an inefficient system has lower excess entropy generation than an efficient system, the prediction is falsified.

7.3 Domain-Specific Predictions

Domain	Prediction	Falsification
Physical	$\kappa\kappa$ correlates with thermal efficiency	$\kappa\kappa$ high but efficiency low
Biological	$\kappa\kappa$ correlates with metabolic efficiency	$\kappa\kappa$ high but metabolic cost high
Cognitive	$\kappa\kappa$ correlates with learning efficiency	$\kappa\kappa$ high but learning cost high
Social	$\kappa\kappa$ correlates with institutional efficiency	$\kappa\kappa$ high but coordination cost high

8. Experimental Design

8.1 Physical Systems

- **System:** Gas in a piston
- **Perturbation:** Compression
- **Measurement:** Excess entropy generation (heat measurement) and recovery time
- **Test:** Correlation between $\kappa\kappa$ and $1/D^\infty 1/D^\infty$

8.2 Biological Systems

- **System:** Cell culture
- **Perturbation:** Nutrient shock
- **Measurement:** Metabolic rate above resting (oxygen consumption) and recovery time
- **Test:** Correlation between $\kappa\kappa$ and metabolic cost

8.3 Cognitive Systems

- **System:** Human participants in a learning task
- **Perturbation:** Prediction error
- **Measurement:** Free energy dissipation above baseline (EEG complexity, pupil dilation) and belief updating rate
- **Test:** Correlation between $\kappa\kappa$ and free energy dissipation

8.4 Social Systems

- **System:** Institutional response to shocks
 - **Perturbation:** Economic or political crisis
 - **Measurement:** Social entropy production above baseline (allostatic load, cortisol, institutional friction) and recovery time
 - **Test:** Correlation between $\kappa\kappa$ and social entropy production
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9. Open Questions

Question	Status	Difficulty
Q1: Uniqueness of $S(x)$	Are there multiple valid entropy functionals for a given domain?	Hard
Q2: Variational principle	Is there a universal variational principle that yields $S(x)$?	Hard
Q3: Social second law	Does $\sigma_{\text{social}} \geq 0$ always hold during recovery?	Very Hard
Q4: Cross-level entropy	How does entropy generation at one level relate to entropy generation at another?	Hard
Q5: Measurement	Can we measure excess entropy generation in cognitive and social systems directly?	Moderate
Q6: Unification	Can all domain-specific entropy functionals be derived from a single universal functional?	Very Hard

10. Conclusion

Every dissipative system maintains its attractor through continuous reconfiguration. Reconfiguration requires work; work generates excess entropy. The recovery rate κ – corrective permeability – is the rate at which a system reconfigures to return to its attractor after perturbation. We have proposed that κ is a measure of excess entropy generation rate.

We developed an abstract persistence cost framework and proved its equivalence to Lyapunov theory. We then identified entropy production as a physical realization of this cost, deriving: $\kappa = \inf_x \delta(x) \int_0^\infty \sigma_{\text{excess}}(\phi^t(x)) dt$

$$(\phi t(x)) dt \delta(x)$$

where $\sigma_{\text{excess}} = \sigma - \sigma_{\text{ss}}$ is the excess entropy production rate above the system's steady-state baseline – thermodynamic entropy for physical systems, metabolic entropy for biological systems, free energy dissipation for cognitive systems, and social entropy production for social systems.

We proposed a unified benchmark: the attractor is the state of minimum entropy generation for that class of system – zero for equilibrium systems, non-zero steady-state for dissipative systems. This provides a unified criterion for identifying attractors across domains: an attractor is a state from which the system cannot reduce its entropy generation further without losing its defining structure or function.

This unifies physical, biological, cognitive, and social systems. In each domain, persistence requires reconfiguration; reconfiguration generates excess entropy; κ measures the entropy cost of that reconfiguration. The framework is grounded in the second law of thermodynamics and non-equilibrium steady-state thermodynamics, not analogy.

Social Application: The framework provides a thermodynamic interpretation of social dynamics: harmony is a low-entropy attractor state; turbulence is a high-entropy state generated by excess dissipation during reconfiguration. The recovery rate κ measures how efficiently a society transitions from turbulence back to harmony – that is, how quickly it reduces its excess entropy production to zero.

11. Limitations

This paper establishes an abstract persistence cost framework with a proposed thermodynamic realization. Several limitations

should be explicitly acknowledged:

1. **Uniqueness.** Entropy production is not proved to be the unique persistence cost. Many positive functionals $C(x)$ satisfy $\nabla D \cdot f = -C$. The identification of entropy production as the canonical cost is a physically motivated hypothesis, not a mathematical theorem.
 2. **Scope.** The framework does not imply that all domains obey thermodynamics literally. The cognitive and social realizations are proposed hypotheses requiring empirical validation.
 3. **Decay assumption.** Exponential decay of excess is a sufficient assumption to ensure finiteness of D^∞ , not a necessary one. Generalization to L^1 integrable decays (e.g., algebraic) is a priority for future work.
 4. **Basin depth.** Basin depth $B = D^\infty(\text{saddle})$ is defined in terms of the persistence cost functional. Its relationship to classical energy barriers is established only for gradient systems.
 5. **Empirical validation.** The predictions of the framework – particularly the inverse relationship between κ and D^∞ – remain to be tested empirically across domains.
 6. **Low-energy attractor benchmark.** The benchmark proposed in §6.3 is a hypothesis, not a derived theorem. For cognitive systems, it risks conflating thermodynamic entropy production with free-energy minimization – distinct principles whose relationship remains open.
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Deriving Corrective Permeability from the Cumulative Deviation Functional; Robert Galida (June 2026) [F]

Abstract

The attractor framework defines κ (corrective permeability) as the rate at which a system returns to its attractor after perturbation. Historically, κ has been treated as an empirical parameter – fitted to data rather than derived from first principles. This paper derives κ from the framework's foundational object: the cumulative deviation functional $D_T(x) = \int_0^T \delta(\phi_t(x)) dt$ $D_\infty(x) = \int_0^\infty \delta(\phi_t(x)) dt$, where $\delta(x) = d(x, A)$.

We define: $\kappa = \inf_{x \in B} \frac{\delta(x)}{D_\infty(x)}$ $\kappa = \inf_{x \in B} \frac{\delta(x)}{D_\infty(x)}$

We prove that for linear systems $x' = -Ax$ with A symmetric positive definite, this definition recovers the slowest eigenvalue $\lambda_{\min}(A)$ – the conventional notion of corrective permeability. We establish a sharp universal persistence bound $D_\infty(x) \leq \delta(x) / \kappa$, show homogeneity and scale invariance of the variational ratio, and demonstrate consistency with Koopman

spectral theory and resolvent poles for finite-dimensional linear systems. A comparison theorem links κ to classical exponential stability constants. A Hamilton-Jacobi-type transport equation for $D^\infty D^\infty$ is derived. A finite-horizon estimator $\kappa_T = \inf_x \delta(x) DT(x) \kappa T = \inf_x \delta(x) DT(x) \delta(x)$ is provided with exponential convergence under explicit assumptions.

The derivation is rigorous for linear systems and testable. Open questions for nonlinear, multiscale, and stochastic systems are identified.

Keywords: corrective permeability, cumulative deviation functional, attractor framework, Koopman operator, trajectory functional

1. Introduction

The attractor framework has been applied across physics, biology, cognition, and social systems. Its central variable – corrective permeability κ – measures the rate at which a system returns to its attractor after perturbation. Historically, κ has been defined empirically as $\kappa = 1/\tau$, where τ is a measured recovery time constant.

This paper derives κ from a single foundational object: the cumulative deviation functional $DT(x)$. Within the present framework, κ is defined variationally rather than introduced as an empirical fitting parameter. We show that κ is a consequence of the trajectory geometry – specifically, the ratio of initial distance to total cumulative deviation.

The derivation is rigorous for linear systems, connects to established theory (Koopman operators, resolvent poles), and provides a finite-horizon estimator for empirical use. Open

questions for nonlinear and stochastic systems are identified.

2. The Cumulative Deviation Functional

Let X be a metric space with distance function $d(\cdot, \cdot)$. Let $\phi_t(x)$ be the flow of a dynamical system starting from state $x \in X$ at time $t=0$. Let $A \subseteq X$ be an attractor set (a compact, invariant set to which trajectories converge). Let B be the basin of attraction of A .

Define the distance from a point to the attractor: $\delta(x) = d(x, A) = \inf_{a \in A} d(x, a)$

Definition 1 (Cumulative Deviation Functional): For a finite horizon $T > 0$, define: $DT(x) = \int_0^T \delta(\phi_t(x)) dt$

For $T \rightarrow \infty$, define: $D^\infty(x) = \int_0^\infty \delta(\phi_t(x)) dt$

Proposition 1 (Finiteness of D^∞): Assume there exist constants $C < \infty$ and $\mu > 0$ such that: $\delta(\phi_t(x)) \leq Ce^{-\mu t} \delta(x)$

for all $x \in B$. Then $D^\infty(x) < \infty$ for every $x \in B$.

Proof: $D^\infty(x) = \int_0^\infty \delta(\phi_t(x)) dt \leq \int_0^\infty Ce^{-\mu t} \delta(x) dt = C\mu^{-1} \delta(x) < \infty$

Properties (from Galida, 2026a):

Property	Statement
Non-negativity	$DT(x) \geq 0$
Monotonicity	$DT_2(x) \geq DT_1(x)$ for $T_2 \geq T_1$

Property	Statement
Additivity	$D_{T+S}(x) = D_T(x) + DS(\phi_T(x))$ $D_{T+S}(x) = D_T(x) + DS(\phi_T(x))$
Instantaneous growth	$dD_T(x) = \delta(\phi_T(x)) dT$ $dD_{T'}(x) = \delta(\phi_{T'}(x)) dT'$
Occupation measure	$D_T(x) = \int \delta(y) d\mu_T(y)$ $D_{T'}(x) = \int \delta(y) d\mu_{T'}(y)$, where $\mu_T, \mu_{T'}$ is the occupation measure

3. Derivation of Corrective Permeability (κ)

3.1 Variational Definition

Definition 2 (Corrective Permeability): $\kappa = \inf_{x \in B} \frac{\delta(x)}{D^\infty(x)}$ $\kappa = \inf_{x \in B} \frac{\delta(x)}{D^\infty(x)}$

Interpretation: κ is the *effective* recovery rate – the smallest ratio of initial distance to total cumulative deviation. It serves as a global measure of the slowest recovery mode in the basin.

Remark on κ : The definition allows $\kappa = 0$ if $D^\infty(x)$ diverges or if the ratio $\delta(x)/D^\infty(x)$ can be made arbitrarily small. Throughout the remainder of this paper, we assume hypotheses (such as the exponential stability in Proposition 1) that guarantee $\kappa > 0$.

Remark on attainment: The infimum in the definition of κ need not be attained; minimizing sequences may exist without a minimizing state. For linear systems, the infimum is attained on the slow eigenspace.

3.2 Homogeneity and Scale Invariance

Theorem 1 (Homogeneity and Scale Invariance): Suppose the flow satisfies $\phi_t(\alpha x) = \alpha \phi_t(x)$ for all t and all $\alpha > 0$, and the distance function satisfies $\delta(\alpha x) = \alpha \delta(x)$.

Then: $D^\infty(\alpha x) = D^\infty(x)$

Proof: $D^\infty(\alpha x) = \int_0^\infty \delta(\phi_t(\alpha x)) dt = \int_0^\infty \delta(\alpha \phi_t(x)) dt = \alpha \int_0^\infty \delta(\phi_t(x)) dt = \alpha D^\infty(x)$

Corollary: For linear systems, the infimum over all $x \neq 0$ reduces to an infimum over the unit sphere: $\kappa = \inf_{\|x\|=1} \delta(x) / D^\infty(x)$

3.3 Sharp Universal Persistence Bound

Theorem 2 (Sharp Universal Persistence Bound): For any $x \in B$, $\kappa D^\infty(x) \leq \delta(x) \leq \kappa^{-1} D^\infty(x)$

Moreover, the constant $1/\kappa$ is optimal: it is the *smallest* constant such that this inequality holds for all x in the basin.

Proof: By definition of κ as the infimum of $\delta(x) / D^\infty(x)$, we have $\delta(x) / D^\infty(x) \geq \kappa$ for all x . Rearranging gives: $D^\infty(x) \leq \delta(x) / \kappa$

Optimality follows from Theorem 3: for the slow eigenvector v_1 , $D^\infty(v_1) = \delta(v_1) / \kappa$, so no smaller constant can work. \square

3.4 Consistency with Linear Systems

Consider a linear system $\dot{x} = -Ax$, with A symmetric positive definite. Let its eigenvalues be $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, with corresponding orthonormal eigenvectors v_1, v_2, \dots, v_n .

The flow is $\phi_t(x) = e^{-At}x$. The attractor is $A = \{0\}$, and the distance to the attractor is $\delta(x) = \|x\|$.

Theorem 3 (Linear Consistency): For $\dot{x} = -Ax$ with A symmetric positive definite, $\inf_{x \neq 0} \|x\| D^\infty(x) = \lambda_{\min}(A)$ and $\inf_{x \neq 0} \|D^\infty(x)\| \|x\| = \lambda_{\min}(A)$.

Proof:

Since A is symmetric positive definite, e^{-At} is symmetric positive definite with eigenvalues $e^{-\lambda_i t}$. Hence its operator norm is $\|e^{-At}\| = e^{-\lambda_1 t}$. For any $x \neq 0$, $D^\infty(x) = \int_0^\infty \|e^{-At}x\| dt \leq \int_0^\infty \|x\| e^{-\lambda_1 t} dt = \|x\| \lambda_1^{-1} D^\infty(x) = \int_0^\infty \|e^{-At}x\| dt \leq \int_0^\infty \|x\| e^{-\lambda_1 t} dt = \lambda_1^{-1} \|x\|$

Therefore: $\|x\| D^\infty(x) \geq \lambda_1^{-1} D^\infty(x) \|x\| \geq \lambda_1^{-1} \|x\|^2$

To show equality is achieved, take $x = v_1$ (the eigenvector corresponding to λ_1). Then: $\|e^{-At}v_1\| = \|v_1\| e^{-\lambda_1 t}$

and: $D^\infty(v_1) = \int_0^\infty \|v_1\| e^{-\lambda_1 t} dt = \|v_1\| \lambda_1^{-1} D^\infty(v_1) = \int_0^\infty \|v_1\| e^{-\lambda_1 t} dt = \lambda_1^{-1} \|v_1\|$

Thus: $\|v_1\| D^\infty(v_1) = \lambda_1^{-1} D^\infty(v_1) \|v_1\| = \lambda_1^{-1} \|v_1\|^2$

Hence: $\inf_{x \neq 0} \|x\| D^\infty(x) = \lambda_1^{-1} \inf_{x \neq 0} \|D^\infty(x)\| \|x\| = \lambda_1^{-1}$

Corollary: For linear systems, the variational definition of κ recovers the slowest eigenvalue – the conventional notion of corrective permeability.

3.5 Transport Equation

Theorem 4 (Transport Equation): Assume the vector field f is C^1 , the flow ϕ_t is C^1 , and D^∞ is continuously differentiable on $B \subset \mathbb{R}^n$. Then: $\nabla D^\infty(x) \cdot f(x) = -\delta(x)$

Proof: From the definition: $D^\infty(\phi_s(x)) = D^\infty(x) - Ds(x) D^\infty(\phi_s(x))$

Differentiating with respect to s at $s=0$: $\frac{d}{ds} D^\infty(\phi_s(x)) \Big|_{s=0} = -\delta(x)$

By the chain rule: $\nabla D^\infty(x) \cdot f(x) = -\delta(x)$

Interpretation: This is a first-order transport equation, $f \cdot \nabla D = -\delta$, which belongs to the broader Hamilton-Jacobi family but lacks a Hamiltonian in the usual sense. It may serve as a foundation for numerical computation and further theoretical development.

3.6 Local vs. Global Interpretation

The variational definition $\kappa = \inf_x \delta(x) D^\infty(x)$ is **global** – it is the slowest recovery rate over the entire basin. This is not necessarily the same as the local recovery rate near the attractor (the slowest eigenvalue of the linearization). For linear systems, they coincide. For nonlinear systems, they may differ if transient excursions produce slower effective recovery than the local linearization predicts.

This distinction is important: κ is a global invariant of the

basin, not merely a local property of the attractor. The relationship between the global κ and the local Lyapunov exponent is an open question (see §6).

3.7 Non-Symmetric Linear Systems

For a general linear system $x' = Ax$ (where A is stable, i.e., all eigenvalues have negative real parts), the same principle holds in the diagonalizable case. The slowest mode corresponds to the eigenvalue with the largest real part (closest to zero).

Conjecture: An analogous result holds for non-normal linear systems under additional assumptions on the semigroup, such as a uniformly exponentially stable semigroup satisfying suitable norm bounds. This remains an open question.

3.8 Comparison with Exponential Stability

Theorem 5 (Comparison with Exponential Stability): Suppose the system satisfies the exponential stability bound: $\delta(\phi_t(x)) \leq C e^{-\mu t} \delta(x)$

for all $x \in B$, with constants $C < \infty$ and $\mu > 0$. Then: $\kappa \geq \mu C$

Proof: From the stability bound: $D^\infty(x) = \int_0^\infty \delta(\phi_t(x)) dt \leq \int_0^\infty C e^{-\mu t} \delta(x) dt = C \mu \delta(x)$

Therefore: $\delta(x) D^\infty(x) \geq \mu C \delta(x) \Rightarrow \kappa \geq \mu C$

Taking the infimum over x : $\kappa = \inf_x \delta(x) D^\infty(x) \geq \mu C \kappa = \mu C \inf_x \delta(x) \Rightarrow \kappa \geq \mu C$

Interpretation: The variational constant κ is bounded below by the exponential stability constant μ/C .

4. Connections to Existing Theory

4.1 Koopman Operator

The Koopman operator K_t acts on observables as: $(K_t f)(x) = f(\phi_t(x))$

For linear systems $\dot{x} = -Ax$, the Koopman eigenvalues are $e^{-\lambda t}$. The dominant nontrivial eigenvalue (largest less than 1) is $e^{-\lambda_1 t}$, corresponding to the slowest decay rate.

For finite-dimensional linear systems, $\rho = e^{-\lambda_{\min} t}$, and therefore: $-\ln \rho = \lambda_{\min} t = \kappa t$

Thus, under the hypotheses of Theorem 3, the variational constant equals the exponential decay rate associated with the dominant Koopman eigenvalue.

4.2 Resolvent Poles

For finite-dimensional stable linear systems, the resolvent $(sI + A)^{-1}$ has poles at $s = -\lambda_i$. The pole closest to the imaginary axis is $s = -\lambda_1$.

Since Theorem 3 identifies $\kappa = \lambda_{\min}$, and the resolvent poles are $s_i = -\lambda_i$, we obtain: $\kappa = \min_i \operatorname{Re}(s_i)$

for finite-dimensional linear systems.

5. Finite-Horizon Estimation

In practice, we can only measure finite trajectories. Define the finite-horizon estimator: $\kappa_T = \inf_{x \in K} \delta(x) D_T(x)$ $\kappa_T = \inf_{x \in K} \delta(x) D_T(x)$

where $K \subset B$ is compact and $K \cap A = \emptyset$.

Proposition 2 (Finite-Horizon Estimation): Assume:

1. The flow $\phi_t(x)$ is jointly continuous in (t, x) .
2. $\delta(x)$ is continuous.
3. The exponential stability bound $\delta(\phi_t(x)) \leq C e^{-\mu t} \delta(x)$ holds uniformly for all $x \in K$, with $\mu > 0$.

Then the variational constant κ (from Definition 2) satisfies $\kappa \geq \mu/C$ by Theorem 5, and: $\kappa_T \rightarrow \kappa$ as $T \rightarrow \infty$

with error: $|\kappa_T - \kappa| = O(e^{-\mu T})$

Proof: For any $x \in K$, the tail bound gives: $|D^\infty(x) - D_T(x)| = \int_T^\infty \delta(\phi_t(x)) dt \leq C e^{-\mu T} \delta(x) \mu |D^\infty(x) - D_T(x)| = \int_T^\infty \delta(\phi_t(x)) dt \leq \mu C e^{-\mu T} \delta(x)$

Since $\delta(x)$ is bounded on the compact set K , let $M = \sup_{x \in K} \delta(x) < \infty$. Then: $|D^\infty(x) - D_T(x)| \leq C M e^{-\mu T} \mu |D^\infty(x) - D_T(x)| \leq \mu C M e^{-\mu T}$

The right-hand side is independent of x and tends to zero as $T \rightarrow \infty$. Hence $D_T \rightarrow D^\infty$ uniformly on K .

Moreover, since K is compact and $K \cap A = \emptyset$, continuity of δ gives $\inf_{x \in K} \delta(x) > 0$. Since $D_T(x)$ is continuous (by assumptions 1-2) and monotonically non-

decreasing in T (from §2), for any fixed finite $T_0 > 0$, $D^\infty(x) \geq DT_0(x) D^\infty(x) \geq DT_0(x)$, and $DT_0(x)$ is continuous and strictly positive on K . A continuous, strictly positive function on a compact set has a positive infimum: $m = \inf_{x \in K} DT_0(x) > 0$.

Thus: $\inf_{x \in K} D^\infty(x) \geq m > 0$.

Uniform convergence of DT_0 to D^∞ on K therefore implies uniform convergence of $\delta(x)/DT_0(x)$ to $\delta(x)/D^\infty(x)$. Consequently, the infima converge. \square

6. Open Questions

Question	Status	Difficulty
Q1: Nonlinear systems	Does $\inf_{x \in K} \delta(x) D^\infty(x) = \kappa$ equal the local Lyapunov exponent?	Hard
Q2: Local vs. global consistency	Does $\lim_{x \rightarrow A} \delta(x) D^\infty(x) = \kappa$ hold for general nonlinear systems?	Hard
Q3: Non-normal systems	Does the infimum equal the slowest eigenvalue for non-normal AA ?	Moderate
Q4: Multiple timescales	Does the infimum isolate the slowest timescale?	Hard
Q5: Stochastic systems	How does noise affect the finite-horizon estimator?	Hard
Q6: Multiple attractors	How does κ behave in basins with multiple attractors?	Moderate

7. Conclusion

This paper derives corrective permeability κ from the cumulative deviation functional $DT(x)DT^\square(x)$. The variational definition: $\kappa = \inf_x \delta(x) D^\infty(x) \kappa = x \inf_x D^\infty^\square(x) \delta(x)$

is shown to recover the slowest eigenvalue for linear systems, consistent with the conventional empirical definition $\kappa = 1/\tau$. A sharp universal persistence bound $D^\infty(x) \leq \delta(x)/\kappa$ $D^\infty^\square(x) \leq \delta(x)/\kappa$ is established. A comparison theorem links κ to classical exponential stability constants. A Hamilton-Jacobi-type transport equation for D^∞ is derived. Connections to Koopman theory and resolvent theory are established for finite-dimensional linear systems. A finite-horizon estimator $\kappa T \kappa T^\square$ is provided with exponential convergence under explicit assumptions.

Key contribution: Within the present framework, κ is defined variationally rather than introduced as an empirical fitting parameter – at least for the class of systems analyzed here.

Next steps: Extend the derivation to nonlinear systems (Q1–Q2), non-normal systems (Q3), multiple timescales (Q4), and stochastic dynamics (Q5).

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