Deriving Corrective Permeability from the Cumulative Deviation Functional; Robert Galida (June 2026) [F]

Abstract

The attractor framework defines κ (corrective permeability) as the rate at which a system returns to its attractor after perturbation. Historically, κ has been treated as an empirical parameter — fitted to data rather than derived from first principles. This paper derives κ from the framework's foundational object: the cumulative deviation functional $D_T(x)=\int_0^T \delta(\phi_t(x))\,dt$, where $\delta(x)=d(x,A)$.

We define: $\kappa=\inf_{x\in B\setminus A}\frac{\delta(x)}{D_\infty(x)}$

We prove that for linear systems $\dot{x}=-Ax$ with A symmetric positive definite, this definition recovers the slowest eigenvalue $\lambda_{\min}(A)$ — the conventional notion of corrective permeability. We establish a sharp universal persistence bound $D_\infty(x)\le\delta(x)/\kappa$, show homogeneity and scale invariance of the variational ratio, and demonstrate consistency with Koopman spectral theory and resolvent poles for finite-dimensional linear systems. A comparison theorem links κ to classical exponential stability constants. A Hamilton-Jacobi-type transport equation for D_∞ is derived. A finite-horizon estimator $\kappa_T=\inf_x\frac{\delta(x)}{D_T(x)}$ is provided with exponential convergence under explicit assumptions.

The derivation is rigorous for linear systems and testable. Open questions for nonlinear, multiscale, and stochastic systems are identified.

Keywords: corrective permeability, cumulative deviation functional, attractor framework, Koopman operator, trajectory functional

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1. Introduction

The attractor framework has been applied across physics, biology, cognition, and social systems. Its central variable — corrective permeability κ — measures the rate at which a system returns to its attractor after perturbation. Historically, κ has been defined empirically as $\kappa=1/\tau$, where τ is a measured recovery time constant.

This paper derives κ from a single foundational object: the cumulative deviation functional $D_T(x)$. Within the present framework, κ is defined variationally rather than introduced as an empirical fitting parameter. We show that κ is a consequence of the trajectory geometry — specifically, the ratio of initial distance to total cumulative deviation.

The derivation is rigorous for linear systems, connects to established theory (Koopman operators, resolvent poles), and provides a finite-horizon estimator for empirical use. Open questions for nonlinear and stochastic systems are identified.

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2. The Cumulative Deviation Functional

Let X be a metric space with distance function $\|\cdot-\cdot\|$. Let $\phi_t(x)$ be the flow of a dynamical system starting from state $x \in X$ at time $t=0$. Let $A \subseteq X$ be an attractor set (a compact, invariant set to which trajectories converge). Let B be the basin of attraction of A.

Define the distance from a point to the attractor:

$$\delta(x) = d(x, A) = \inf_{a \in A} \|x - a\|$$

Definition 1 (Cumulative Deviation Functional): For a finite horizon $T>0$, define:

$$D_T(x) = \int_0^T \delta(\phi_t(x))\, dt$$

For $T \to \infty$, define:

$$D_\infty(x) = \int_0^\infty \delta(\phi_t(x))\, dt$$

Proposition 1 (Finiteness of D_∞): Assume there exist constants $C<\infty$ and $\mu>0$ such that:

$$\delta(\phi_t(x)) \leq C e^{-\mu t} \delta(x)$$

for all $x \in B$. Then $D_\infty(x) < \infty$ for every $x \in B$.

Proof:

$$D_\infty(x) = \int_0^\infty \delta(\phi_t(x))\, dt \leq \int_0^\infty C e^{-\mu t} \delta(x)\, dt = \frac{C}{\mu} \delta(x) < \infty \quad \square$$

Properties (from Galida, 2026a):

Property	Statement
Non-negativity	$D_T(x) \geq 0$
Monotonicity	$D_{T_2}(x) \geq D_{T_1}(x)$ for $T_2 \geq T_1$
Additivity	$D_{T+S}(x) = D_T(x) + D_S(\phi_T(x))$
Instantaneous growth	$\frac{d}{dT} D_T(x) = \delta(\phi_T(x))$

Property	Statement
Occupation measure	$D_T(x)=\int \delta(y)\, d\mu_T(y)$, where μ_T is the occupation measure

3. Derivation of Corrective Permeability (κ)

3.1 Variational Definition

Definition 2 (Corrective Permeability): $\kappa=\inf_{x\in B\setminus A}\frac{\delta(x)}{D_\infty(x)}$

Interpretation: κ is the *effective* recovery rate — the smallest ratio of initial distance to total cumulative deviation. It serves as a global measure of the slowest recovery mode in the basin.

Remark on κ: The definition allows $\kappa=0$ if $D_\infty(x)$ diverges or if the ratio $\delta(x)/D_\infty(x)$ can be made arbitrarily small. Throughout the remainder of this paper, we assume hypotheses (such as the exponential stability in Proposition 1) that guarantee $\kappa>0$.

Remark on attainment: The infimum in the definition of κ need not be attained; minimizing sequences may exist without a minimizing state. For linear systems, the infimum is attained on the slow eigenspace.

3.2 Homogeneity and Scale Invariance

Theorem 1 (Homogeneity and Scale Invariance): Suppose the flow satisfies $\phi_t(\alpha x)=\alpha\phi_t(x)$ for all t and all $\alpha>0$, and the distance function satisfies $\delta(\alpha x)=\alpha\delta(x)$. Then:$$\frac{D_\infty(\alpha x)}{\delta(\alpha x)}=\frac{D_\infty(x)}{\delta(x)}$$

Proof:$$D_\infty(\alpha x)=\int_0^\infty \delta(\phi_t(\alpha x))\,dt=\int_0^\infty \delta(\alpha\phi_t(x))\,dt=\alpha\int_0^\infty \delta(\phi_t(x))\,dt=\alpha D_\infty(x)$$

Corollary: For linear systems, the infimum over all $x\neq 0$ reduces to an infimum over the unit sphere:$$\kappa=\inf_{\|x\|=1}\frac{\delta(x)}{D_\infty(x)}$$

3.3 Sharp Universal Persistence Bound

Theorem 2 (Sharp Universal Persistence Bound): For any $x\in\mathcal{B}_A$:$$D_\infty(x)\le\frac{\delta(x)}{\kappa}$$

Moreover, the constant $1/\kappa$ is optimal: it is the *smallest* constant such that this inequality holds for all x in the basin.

Proof: By definition of κ as the infimum of $\delta(x)/D_\infty(x)$, we have $\delta(x)/D_\infty(x)\ge\kappa$ for all x. Rearranging gives:$$D_\infty(x)\le\frac{\delta(x)}{\kappa}$$

Optimality follows from Theorem 3: for the slow eigenvector v_1, $D_\infty(v_1)=\delta(v_1)/\kappa$, so no smaller constant can work.$\square$

3.4 Consistency with Linear Systems

Consider a linear system $\dot{x}=-Ax$, with A symmetric positive definite. Let its eigenvalues be $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n$, with corresponding orthonormal eigenvectors $v_1,v_2,\dots,v_n$.

The flow is $\phi_t(x)=e^{-At}x$. The attractor is $A=\{0\}$, and the distance to the attractor is $\delta(x)=\|x\|$.

Theorem 3 (Linear Consistency): For $\dot{x}=-Ax$ with A symmetric positive definite,

$$\inf_{x\neq 0}\frac{\|x\|}{D_\infty(x)}=\lambda_{\min}(A)$$

Proof:

Since A is symmetric positive definite, e^{-At} is symmetric positive definite with eigenvalues $e^{-\lambda_i t}$. Hence its operator norm is $\|e^{-At}\|=e^{-\lambda_1 t}$. For any $x\neq 0$:

$$D_\infty(x)=\int_0^\infty \|e^{-At}x\|\,dt \le \int_0^\infty \|x\|e^{-\lambda_1 t}\,dt=\frac{\|x\|}{\lambda_1}$$

Therefore:

$$\frac{\|x\|}{D_\infty(x)}\ge \lambda_1$$

To show equality is achieved, take $x=v_1$ (the eigenvector corresponding to λ_1). Then:

$$\|e^{-At}v_1\|=\|v_1\|e^{-\lambda_1 t}$$

and:

$$D_\infty(v_1)=\int_0^\infty \|v_1\|e^{-\lambda_1 t}\,dt=\frac{\|v_1\|}{\lambda_1}$$

Thus:

$$\frac{\|v_1\|}{D_\infty(v_1)}=\lambda_1$$

Hence:

$$\inf_{x\neq 0}\frac{\|x\|}{D_\infty(x)}=\lambda_1 \quad \square$$

Corollary: For linear systems, the variational definition of κ recovers the slowest eigenvalue — the conventional notion of corrective permeability.

3.5 Transport Equation

Theorem 4 (Transport Equation): Assume the vector field f is C^1, the flow ϕ_t is C^1, and D_∞ is continuously differentiable on $B \setminus A$.

Then: $\nabla D_\infty(x)\cdot f(x) = -\delta(x)$

Proof: From the definition: $D_\infty(\phi_s(x)) = D_\infty(x) - Ds(x)$

Differentiating with respect to s at $s=0$: $\frac{d}{ds}D_\infty(\phi_s(x))\Big|_{s=0} = -\delta(x)$

By the chain rule: $\nabla D_\infty(x)\cdot f(x) = -\delta(x)$ $\square$

Interpretation: This is a first-order transport equation, $f\cdot\nabla D = -\delta$, which belongs to the broader Hamilton-Jacobi family but lacks a Hamiltonian in the usual sense. It may serve as a foundation for numerical computation and further theoretical development.

3.6 Local vs. Global Interpretation

The variational definition $\kappa = \inf_x \frac{\delta(x)}{D_\infty(x)}$ is **global** — it is the slowest recovery rate over the entire basin. This is not necessarily the same as the local recovery rate near the attractor (the slowest eigenvalue of the linearization). For linear systems, they coincide. For nonlinear systems, they may differ if transient excursions produce slower effective recovery than the local linearization predicts.

This distinction is important: κ is a global invariant of the basin, not merely a local property of the attractor. The relationship between the global κ and the local Lyapunov exponent is an open question (see §6).

3.7 Non-Symmetric Linear Systems

For a general linear system $\dot{x} = Ax$ (where A is stable, i.e., all eigenvalues have negative real parts), the same principle holds in the diagonalizable case. The slowest mode corresponds to the eigenvalue with the largest real part (closest to zero).

Conjecture: An analogous result holds for non-normal linear systems under additional assumptions on the semigroup, such as a uniformly exponentially stable semigroup satisfying suitable norm bounds. This remains an open question.

3.8 Comparison with Exponential Stability

Theorem 5 (Comparison with Exponential Stability): Suppose the system satisfies the exponential stability bound:$$\delta(\phi_t(x)) \leq Ce^{-\mu t}\delta(x)$$

for all $x \in B$, with constants $C < \infty$ and $\mu > 0$. Then:$$\kappa \geq \frac{\mu}{C}$$

Proof: From the stability bound:$$D_\infty(x) = \int_0^\infty \delta(\phi_t(x))\,dt \leq \int_0^\infty Ce^{-\mu t}\delta(x)\,dt = \frac{C}{\mu}\delta(x)$$

Therefore:$$\frac{\delta(x)}{D_\infty(x)} \geq \frac{\mu}{C}$$

Taking the infimum over x:$$\kappa = \inf_x \frac{\delta(x)}{D_\infty(x)} \geq \frac{\mu}{C} \quad \square$$

Interpretation: The variational constant κ is bounded below by the exponential stability constant μ/C.

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4. Connections to Existing Theory

4.1 Koopman Operator

The Koopman operator K^t acts on observables as:$$(K^t f)(x) = f(\phi^t(x))$$

For linear systems $\dot{x} = -Ax$, the Koopman eigenvalues are $e^{-\lambda_i t}$. The dominant nontrivial eigenvalue (largest less than 1) is $e^{-\lambda_1 t}$, corresponding to the slowest decay rate.

For finite-dimensional linear systems, $\rho = e^{-\lambda_{\min} t}$, and therefore:$$-\frac{1}{t}\log \rho = \lambda_{\min} = \kappa$$

Thus, under the hypotheses of Theorem 3, the variational constant equals the exponential decay rate associated with the dominant Koopman eigenvalue.

4.2 Resolvent Poles

For finite-dimensional stable linear systems, the resolvent $(sI + A)^{-1}$ has poles at $s = -\lambda_i$. The pole closest to the imaginary axis is $s = -\lambda_1$.

Since Theorem 3 identifies $\kappa = \lambda_{\min}$, and the resolvent poles are $s_i = -\lambda_i$, we obtain:$$\kappa = \min_i |\Re(s_i)|$$

for finite-dimensional linear systems.

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5. Finite-Horizon Estimation

In practice, we can only measure finite trajectories. Define the finite-horizon estimator:

$$\kappa_T = \inf_{x \in K} \frac{D_T(x)}{\delta(x)}$$

where $K \subset B$ is compact and $K \cap A = \emptyset$.

Proposition 2 (Finite-Horizon Estimation): Assume:

1. The flow $\phi_t(x)$ is jointly continuous in (t, x).
2. $\delta(x)$ is continuous.
3. The exponential stability bound $\delta(\phi_t(x)) \le C e^{-\mu t} \delta(x)$ holds uniformly for all $x \in K$, with $\mu > 0$.

Then the variational constant κ (from Definition 2) satisfies $\kappa \ge \mu / C$ by Theorem 5, and:

$$\kappa_T \to \kappa \text{ as } T \to \infty$$

with error:

$$|\kappa_T - \kappa| = O(e^{-\mu T})$$

Proof: For any $x \in K$, the tail bound gives:

$$|D_\infty(x) - D_T(x)| = \int_T^\infty \delta(\phi_t(x))\, dt \le \frac{C e^{-\mu T} \delta(x)}{\mu}$$

Since $\delta(x)$ is bounded on the compact set K, let $M = \sup_{x \in K} \delta(x) < \infty$. Then:

$$|D_\infty(x) - D_T(x)| \le \frac{C M e^{-\mu T}}{\mu}$$

The right-hand side is independent of x and tends to zero as $T \to \infty$. Hence $D_T \to D_\infty$ uniformly on K.

Moreover, since K is compact and $K \cap A = \emptyset$, continuity of δ gives $\inf_{x \in K} \delta(x) > 0$. Since $D_T(x)$ is continuous (by assumptions 1–2) and monotonically non-decreasing in T (from §2), for any fixed finite $T_0 > 0$, $D_\infty(x) \ge D_{T_0}(x)$, and D_{T_0} is continuous and strictly positive on K. A continuous, strictly positive function on a compact set has a positive infimum: $m = \inf_{x \in K} D_{T_0}(x) > 0$

Thus: $\inf_{x \in K} D_\infty(x) \geq m > 0$

Uniform convergence of D_T to D_∞ on K therefore implies uniform convergence of $\delta(x)/D_T(x)$ to $\delta(x)/D_\infty(x)$. Consequently, the infima converge. $\square$

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6. Open Questions

Question	Status	Difficulty
Q1: Nonlinear systems	Does $\inf \frac{\delta}{D_\infty}$ equal the local Lyapunov exponent?	Hard
Q2: Local vs. global consistency	Does $\lim_{x \to A} \frac{\delta(x)}{D_\infty(x)} = \kappa$ hold for general nonlinear systems?	Hard
Q3: Non-normal systems	Does the infimum equal the slowest eigenvalue for non-normal A?	Moderate
Q4: Multiple timescales	Does the infimum isolate the slowest timescale?	Hard
Q5: Stochastic systems	How does noise affect the finite-horizon estimator?	Hard
Q6: Multiple attractors	How does κ behave in basins with multiple attractors?	Moderate

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7. Conclusion

This paper derives corrective permeability κ from the cumulative deviation functional $D_T(x)$. The variational definition:$\kappa = \inf_x \frac{\delta(x)}{D_\infty(x)}$

is shown to recover the slowest eigenvalue for linear systems, consistent with the conventional empirical definition $\kappa = 1/\tau$. A sharp universal persistence bound $D_\infty(x) \le \delta(x)/\kappa$ is established. A comparison theorem links κ to classical exponential stability constants. A Hamilton-Jacobi-type transport equation for D_∞ is derived. Connections to Koopman theory and resolvent theory are established for finite-dimensional linear systems. A finite-horizon estimator κ_T is provided with exponential convergence under explicit assumptions.

Key contribution: Within the present framework, κ is defined variationally rather than introduced as an empirical fitting parameter — at least for the class of systems analyzed here.

Next steps: Extend the derivation to nonlinear systems (Q1–Q2), non-normal systems (Q3), multiple timescales (Q4), and stochastic dynamics (Q5).

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The Persistence Functional: A Candidate Formal Foundation for the Attractor Framework; Robert Galida (July 2026) [F]

Abstract

The attractor framework provides a domain-general vocabulary for describing persistence and change across physical, biological, cognitive, and social systems. However, its core variables—κ (corrective permeability), B (basin depth), and R (reality alignment)—have been defined inconsistently across application papers, and their formal relationships have remained implicit. This paper proposes a candidate mathematical formalization for the framework.

The central mathematical innovation of this paper is treating persistence as a functional defined over trajectories—$D_T(x)=\int_0^T d(\phi_\tau(x),A)\,d\tau$—rather than as a scalar property of states. We prove several mathematical properties of D_T, including non-negativity, monotonicity in T, additivity, Lipschitz continuity with respect to initial conditions, and a bound relating D_∞ to the recovery rate κ: $D_\infty(x)\leq\frac{C}{\kappa}d(x,A)$. We establish connections to dynamic programming and ergodic theory via occupation measures. We introduce a complementary **topological persistence functional** $P_{\text{topo}}(t)$, which measures the lifetime of topological features in the trajectory's state-space geometry, and the **topological evolution rate** $E(t)$.

We unify the framework's variable set: κ is the recovery rate (operationalized as $1/\tau$); γ is a proposed drift rate for persistent chaos, grounded in the literature on high-dimensional neural networks; B is the energy barrier (basin depth); $\tilde{B}$ is a complementary persistence depth; R is the expected log predictive likelihood. We propose testable predictions linking $E(t)$ to κ and γ, and provide a falsifiable experimental protocol using neural network training and persistent homology.

The paper offers a candidate formal foundation, with explicit definitions, mathematical properties, and empirical grounding. All unverified sources are clearly labeled as such.

Keywords: attractor framework, persistence functional, cumulative deviation, topological persistence, corrective permeability, basin depth, reality alignment, persistent homology

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1. Introduction

The attractor framework has been applied across physics (hydrogen decay, Jeans instability), biology (ECM mechanics, HRV), cognition (belief updating, performance attractors), and social systems (religious attractors, civilizational dynamics). A common vocabulary has emerged: κ (corrective permeability), B (basin depth), and R (reality alignment). However, these variables have been defined inconsistently across papers, and their formal relationships have remained implicit. This paper proposes a candidate mathematical formalization that addresses these inconsistencies.

The central mathematical innovation of this paper is treating persistence as a functional defined over trajectories rather than as a scalar property of states. $D_T(x)=\int_0^T d(\phi_\tau(x),A)\,d\tau$ can be understood as a type of **action functional** (carefully qualified). Like the classical action $\int L(q,\dot{q})\,dt$, it assigns a scalar to an entire trajectory, is additive under concatenation, and suggests variational and optimal-control interpretations. However, it is not the mechanical action; it is a cumulative deviation functional that measures time away from equilibrium. This moves the framework into the domain of trajectory-level analysis, aligning it with modern dynamical systems and geometric control theory.

We introduce the **cumulative deviation functional** $D_T(x)$ as this central object, and we establish its mathematical properties, including its relationship to the recovery rate κ. We introduce a complementary **topological persistence functional** $P_{topo}(t)$ and the **topological evolution rate** $E(t)$. We unify the framework's variable set with operational definitions and propose testable predictions with falsification criteria.

1.1 Scope and Status

This paper is a **candidate formalization**—it provides definitions, mathematical properties, and empirical hypotheses. It is not a completed empirical validation; that is the subject of future work. All claims are labeled as **definitions** (part of the formal structure), **propositions/theorems** (proved), **hypotheses** (testab le predictions), or **heuristics** (suggestive connections not yet formalized). This distinction is maintained throughout.

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2. Formal Definitions

Let X be a metric space with distance function $\|\cdot - \cdot\|$. Let $\phi_\tau(x)$ be the flow of a dynamical system starting from state $x \in X$ at time $\tau = 0$. Let $A \subseteq X$ be an attractor set (a compact, invariant set to which trajectories converge).

Assume the flow is continuous and measurable so that $d(\phi_\tau(x), A)$ is measurable. The flow ϕ_τ satisfies the semigroup property $\phi_{t+s} = \phi_t \circ \phi_s$ for all $t, s \geq 0$, with $\phi_0 = \mathrm{id}$. We assume $d(\phi_\tau(x), A) \in L^1([0,T])$ for all finite T, so the integral defining D_T is well-defined.

Define the distance from a point to the attractor:$$d(x, A) = \inf_{a \in A} \|x - a\|$$

The definition applies to any metric space; for infinite-dimensional spaces, the usual measurability and integrability conditions are assumed.

2.1 Cumulative Deviation Functional

Definition 1 (Cumulative Deviation Functional): For a finite horizon $T > 0$, the cumulative deviation functional is:$$D_T(x) = \int_0^T d(\phi_\tau(x), A)\, d\tau$$

Interpretation: $D_T(x)$ is the total accumulated deviation from the attractor over the interval $[0, T]$. It measures integrated error, residence-time-weighted distance, or accumulated regret. This is **not** a path length; it measures time spent away from equilibrium, whereas path length $\int \|\dot{\phi}_\tau(x)\|\, d\tau$ measures distance traveled.

Domain generality: This definition applies to any system with a well-defined state space, a flow, and an attractor set. It does not require linearity, differentiability, or specific functional forms.

Empirical note: D_T is the fundamental object for empirical work; D_∞ is primarily an analytical limit used for theoretical bounds.

Note: D_T is not a Lyapunov function. A Lyapunov function is a scalar function of the current state; D_T is a functional of the entire trajectory. It does not decrease monotonically along trajectories, and it does not provide pointwise stability information. Its purpose is to measure accumulated history, not instantaneous energy.

Occupation measure connection: Define the occupation measure of the trajectory up to time T as:

$$\mu_T(B)=\int_0^T \mathbf{1}_B(\phi_\tau(x))\,d\tau$$

for measurable $B \subseteq X$. Then:

$$D_T(x)=\int_X d(y,A)\,d\mu_T(y)$$

Thus D_T is the expected distance to the attractor under the occupation measure. This connects the functional directly to ergodic theory and occupation measure analysis. For foundational treatments of occupation measures and invariant measures, see Ruelle (1989) and Bowen (1975).

2.1.1 Why the L¹ Trajectory Functional?

The choice of the L¹ integral over alternatives is motivated by the following properties:

- **Linearity:** Each moment contributes equally; accumulation is additive over time.
- **Physical units:** For systems with a natural distance metric, D_T has units of distance × time, which is interpretable as accumulated deviation.
- **Simplicity:** It is the simplest nontrivial trajectory functional that is not a path length.
- **Analogy:** It mirrors cumulative regret and occupation measures in control theory and ergodic theory.
- **Avoidance of overweighting:** Unlike d^2, it does not disproportionately weight large deviations; unlike max, it is sensitive to the full trajectory.

This is one natural choice; other functionals (e.g., d^p, exponentially weighted integrals) could be substituted without changing the framework’s structure.

2.2 Topological Persistence Functional

Let $X_\tau=\{\phi_s(x):s\in[0,\tau]\}$ be the trajectory segment up to time τ. Let $PH_k(X_\tau)$ be the k-dimensional persistent homology of the point cloud X_τ at scale ϵ. Each feature (component, loop, void) has a birth scale b and a death scale d, with persistence $d-b$. For foundational treatments of persistent homology, see Edelsbrunner & Harer (2010) or Carlsson (2009).

Definition 2 (Topological Persistence Functional): We define the following complementary topological persistence functional.
For $t\geq 0$: $P_{\text{topo}}(t)=\int_0^t \sum_{k\geq 0} \sum_{(b,d)\in PH_k(X_\tau)} (d-b)\, d\tau$

The map $\tau \mapsto PH_k(X_\tau)$ is piecewise constant on intervals where the trajectory does not cross a homology-critical threshold. Assuming the trajectory crosses such thresholds at discrete times, the integral is well-defined as a sum of piecewise continuous segments. This is the standard assumption in time-varying persistent homology (see Carlsson & Zomorodian, 2009).

Interpretation: $P_{\text{topo}}(t)$ is the total lifetime of all topological features in the trajectory’s state-space geometry up to time t. This is a separate mathematical object from D_T; the relationship between them is an empirical hypothesis. This is one possible choice among several topological summaries (e.g., persistence landscapes, persistence images) and is selected because it mirrors the cumulative interpretation of D_T, rather than because it is uniquely canonical. Other stable summaries—such as persistence landscapes, persistence images, or Betti curves—could be substituted for the present functional without changing the framework's structure.

Measurement: In practice, $P_{\text{topo}}(t)$ is computed by sampling the trajectory at discrete times, computing persistent homology on latent activation manifolds, and summing the persistence of all features using standard libraries (e.g., GUDHI, Ripser). Turner & Barak (2023) demonstrated that trained RNNs develop attractors sequentially during training; the topological structure of these attractors can be analyzed using persistent homology.

Falsification: If persistent homology features do not correlate with any behavioral or dynamical measure in a given system, P_{topo} is not a useful construct for that domain.

2.3 Topological Evolution Rate

Definition 3 (Topological Evolution Rate): For a learning system with time-dependent topological persistence, the topological evolution rate is defined as: $E(t) = \frac{d}{dt} P_{\text{topo}}(t)$

where differentiable, and experimentally as $E(t) \approx \frac{\Delta P_{\text{topo}}}{\Delta t}$ over finite intervals.

Interpretation: $E(t)$ measures how quickly the system's topological complexity changes during learning. Negative $E(t)$ indicates topological simplification (compression); positive $E(t)$ indicates increasing complexity (expansion); $E(t) \approx 0$ indicates stagnation. Learning is one possible cause of topological change; random drift, noise, or chaotic wandering can also change topology.

Empirical anchor: Karuppiah, Nazreen Banu et al. (2026)

examine the evolution of topological signatures during training. Turner & Barak (2023) show that RNNs develop attractors sequentially, which may correspond to phases of topological simplification. We hypothesize that successful learning corresponds to negative average values of $E(t)$ over defined phases, but this is a testable claim, not a definition.

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3. Mathematical Properties of the Cumulative Deviation Functional

This section establishes the mathematical behavior of D_T, providing the foundation for its use in the framework.

3.1 Non-negativity

Proposition 1 (Non-negativity): For any $x \in X$ and any $T \geq 0$: $D_T(x) \geq 0$

with equality iff $\phi_\tau(x) \in A$ for almost all $\tau \in [0, T]$.

Proof: The integrand is a distance function $d(\phi_\tau(x), A)$, which is non-negative by definition. The integral of a non-negative function is non-negative. Equality holds only if the integrand is zero almost everywhere.

3.2 Monotonicity in T

Proposition 2 (Monotonicity): For fixed x, $D_T(x)$ is monotonically non-decreasing in T: $D_{T_2}(x) \geq D_{T_1}(x) \text{ for } T_2 \geq T_1$

Proof: For $T_2 \geq T_1$: $D_{T_2}(x) = \int_0^{T_1} d(\phi_\tau(x), A)\, d\tau + \int_{T_1}^{T_2} d(\phi_\tau(x), A)\, d\tau$

The second integral is non-negative by Proposition 1. Therefore $D_{T_2}(x) \geq D_{T_1}(x)$.

Corollary: If the trajectory converges exactly to the attractor at time $\tau_0 < T$, then: $D_T(x) = D_{\tau_0}(x)$ for all $T \geq \tau_0$

3.3 Additivity

Proposition 3 (Additivity): For any $T, S \geq 0$: $D_{T+S}(x) = D_T(x) + D_S(\phi_T(x))$

Proof: $D_{T+S}(x) = \int_0^{T+S} d(\phi_\tau(x), A)\, d\tau = \int_0^T d(\phi_\tau(x), A)\, d\tau + \int_T^{T+S} d(\phi_\tau(x), A)\, d\tau = D_T(x) + \int_0^S d(\phi_{\tau+T}(x), A)\, d\tau = D_T(x) + \int_0^S d(\phi_\tau(\phi_T(x)), A)\, d\tau$ (by the semigroup property) $= D_T(x) + D_S(\phi_T(x))$

This connects D_T naturally to Bellman equations, dynamic programming, and occupation measures.

3.4 Heuristic Connection: Dynamic Programming

The additivity property $D_{T+S}(x) = D_T(x) + D_S(\phi_T(x))$ suggests a natural connection to dynamic programming. For a controlled system $\dot{X} = f(X, u)$ with control $u \in U$, the value function $V(x) = \inf_u D_\infty(x)$

$D_\infty(x)$ would formally satisfy the Hamilton-Jacobi-Bellman equation: $0=\inf_u\{d(x,A)+\nabla V(x)\cdot f(x,u)\}$

This is a standard result for additive cost functionals. A full derivation for the specific functional D_T is left for future work. This section is a heuristic connection, not a formal result.

3.5 Lipschitz Continuity with Respect to Initial Conditions

Proposition 4 (Lipschitz Continuity of D_T): Suppose the flow ϕ_τ is Lipschitz continuous in x with constant L, i.e., $\|\phi_\tau(x)-\phi_\tau(y)\|\le e^{L\tau}\|x-y\|$. Then for any x,y in the basin of A: $|D_T(x)-D_T(y)|\le\int_0^T e^{L\tau}\,d\tau\,\|x-y\|=\frac{e^{LT}-1}{L}\|x-y\|$

Proof: First, note that the distance function $d(\cdot,A)$ is 1-Lipschitz: for any $x,y\in X$, $|d(x,A)-d(y,A)|\le\|x-y\|$

This follows from the triangle inequality and the definition of the infimum. Then, using the Lipschitz property of the flow: $|D_T(x)-D_T(y)|\le\int_0^T|d(\phi_\tau(x),A)-d(\phi_\tau(y),A)|\,d\tau\le\int_0^T\|\phi_\tau(x)-\phi_\tau(y)\|\,d\tau\le\int_0^T e^{L\tau}\|x-y\|\,d\tau=\frac{e^{LT}-1}{L}\|x-y\|$ □

Interpretation: This proposition guarantees that empirical estimates of D_T are robust under small perturbations of initial conditions and establishes that D_T defines a continuous functional on the basin of attraction. This is essential for numerical estimation and experimental measurement.

3.6 Instantaneous Growth Rate

Remark 1 (Instantaneous Growth Rate): If the integrand $d(\phi_\tau(x), A)$ is continuous in τ, then: $$\frac{d}{dT} D_T(x) = d(\phi_T(x), A)$$

This follows directly from the Fundamental Theorem of Calculus.

3.7 Ergodic Limit

Proposition 5 (Ergodic Limit): Suppose the normalized occupation measure $\nu_T = \mu_T / T$ converges weakly to an invariant probability measure μ as $T \to \infty$. Then: $$\lim_{T\to\infty} \frac{1}{T} D_T(x) = \int_X d(y, A)\, d\mu(y)$$

Proof: From the occupation measure representation $D_T(x) = \int d(y, A)\, d\mu_T(y) = T \int d(y, A)\, d\nu_T(y)$, weak convergence of ν_T to μ and boundedness/continuity of $d(\cdot, A)$ gives the result.

This is the pointwise ergodic theorem applied to the observable $d(\cdot, A)$. For the ergodic theory of dynamical systems, see Bowen (1975) and Ruelle (1989).

3.8 Bound under Exponential Stability

Theorem 2 (Bound under Exponential Stability): Suppose the flow $\phi_\tau(x)$ converges to the attractor A with exponential rate $\kappa>0$: $d(\phi_\tau(x),A)\le Ce^{-\kappa\tau}d(x,A)$

for some constant $C<\infty$, for all $\tau\ge 0$. Then: $D_\infty(x)=\int_0^\infty d(\phi_\tau(x),A)\,d\tau\le \frac{C}{\kappa}d(x,A)$

Proof: $D_\infty(x)=\int_0^\infty d(\phi_\tau(x),A)\,d\tau\le\int_0^\infty Ce^{-\kappa\tau}d(x,A)\,d\tau=Cd(x,A)\int_0^\infty e^{-\kappa\tau}\,d\tau=\frac{C}{\kappa}d(x,A)$

Corollary: For linearly stable systems with recovery rate κ, $D_\infty(x)\le\frac{1}{\kappa}d(x,A)$ (when $C=1$).

Important: Exponential stability implies $D_\infty<\infty$. The converse is not claimed; polynomial convergence can also yield finite D_∞.

3.9 Recovery Rate Bound

Corollary 1 (Recovery Rate Bound): For a system satisfying the exponential stability hypothesis with constant C, the recovery rate κ satisfies: $\kappa\le\frac{C\,d(x,A)}{D_\infty(x)}$

For systems with $C=1$ (e.g., normal/symmetric linearizations with no transient overshoot), this reduces to: $\kappa\le\frac{d(x,A)}{D_\infty(x)}$

Proof: From Theorem 2, we have $D_\infty(x)\le\frac{C}{\kappa}d(x,A)$. Rearranging gives $\kappa\le\frac{C\,d(x,A)}{D_\infty(x)}$. When $C=1$, this reduces to $\kappa\le\frac{d(x,A)}{D_\infty(x)}$.

Interpretation: Small cumulative deviation implies rapid recovery (large κ). Large cumulative deviation implies slow recovery (small κ). This formalizes the intuitive link between D_T and κ. The C factor accounts for possible transient overshoot in non-normal systems.

3.10 Finite Horizon Approximation

Proposition 6 (Finite Horizon): For any $\epsilon>0$, there exists a finite T_ϵ such that for all $T>T_\epsilon$: $|D_T(x) - D_\infty(x)| \leq \epsilon$

Proof: This follows directly from Theorem 2 under the exponential stability hypothesis. Since the integrand decays exponentially, the tail integral $\int_T^\infty d(\phi_\tau(x), A)\, d\tau$ can be made arbitrarily small by choosing T sufficiently large.

3.11 Summary of Properties

Property	Statement
Non-negativity	$D_T(x) \geq 0$
Monotonicity	$D_{T_2}(x) \geq D_{T_1}(x)$ for $T_2 \geq T_1$
Additivity	$D_{T+S}(x) = D_T(x) + D_S(\phi_T(x))$
Lipschitz continuity	(| D_T(x) − D_T(y) | \leq \frac{e^{LT} − 1}{L} |x − y|)
Instantaneous growth	$\frac{d}{dT} D_T(x) = d(\phi_T(x), A)$
Ergodic limit	$\lim_{T\to\infty} \frac{1}{T} D_T(x) = \int d(y, A)\, d\mu(y)$

Property	Statement	
Exponential stability implies finite D_∞	$D_\infty(x) \le \frac{C}{\kappa} d(x, A)$	
Recovery bound (general)	$\kappa \le \frac{C\, d(x,A)}{D_\infty(x)}$	
Recovery bound (C=1)	$\kappa \le \frac{d(x,A)}{D_\infty(x)}$	
Finite horizon approximation	$D_T(x) \to D_\infty(x)$ as $T \to \infty$	

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4. The Unified Variable Set

The following variables are defined operationally. Where a variable is a proposal, that is stated explicitly.

4.1 Corrective Permeability (κ)

Definition 4 (Corrective Permeability): κ is the recovery rate of the system to its attractor after a small perturbation. Operationally estimated as $\kappa = 1/\tau$ under approximately exponential relaxation, where τ is the characteristic recovery time constant. This coincides with the exponential convergence exponent in the linearized regime and is consistent with the original definition in the attractor framework.

Relationship to D_T: From Corollary 1, for a system with initial deviation $d(x, A)$, $\kappa \le \frac{C\, d(x,A)}{D_\infty(x)}$.

Note on κ's status: In this paper, κ is treated as a primitive empirical regime parameter. A stronger theory would derive κ from D_T and system geometry; this remains an open direction for future work.

4.2 Drift Rate (γ) — A Proposed Distinction

Definition 5 (Drift Rate): We propose the following operational distinction between dynamical regimes, based on the dominant Lyapunov exponent $\lambda_{\max}$:

Regime	$\lambda_{\max}$	κ	γ	Behavior
Stable attractor	<-0.01	>0	0	Converges to fixed point
Persistent chaos	≈ 0	≈ 0	>0	Wanders without convergence
Full chaos	>0	undefined	>0	Diverges

Thresholds: $\lambda_{\max}<-0.01$, $|\lambda_{\max}|\leq 0.01$, and $\lambda_{\max}>0.01$ (pre-registered, measured in units of 1/epoch). These numerical thresholds are illustrative defaults rather than theoretically privileged constants.

Grounding: This distinction is inspired by the literature on chaos in high-dimensional neural networks (Engelken, Wolf & Abbott, 2023; Sompolinsky, Crisanti & Sommers, 1988; Clark, Abbott & Litwin-Kumar, 2023; Fournier & Urbani, 2023). For the treatment of stochastic and random perturbations, see Arnold (1998).

Falsification: If κ and γ are perfectly correlated (i.e., systems with small κ always have small γ), the distinction is not useful.

4.3 Basin Depth (B) and Persistence

Depth ($\tilde{B}$)

Definition 6a (Basin Depth — Energy Barrier): B is the energy barrier required to escape the basin, measured as the potential difference between the attractor and the saddle point on the basin boundary:

$$B = V(\text{saddle}) - V(\text{attractor})$$

This preserves the original definition from earlier papers.

Definition 6b (Persistence Depth): As a complementary measure, we define:

$$\tilde{B} = \min_{x \in \partial B} D_T(x)$$

This is the cumulative deviation required to reach the basin boundary. The relationship between B and $\tilde{B}$ remains an open mathematical question.

Operational alternative: In practice, the basin boundary may not be well-defined. Estimate B via the Arrhenius relationship $P_{escape} \propto e^{-B/T}$, where T is the noise level.

4.4 Reality Alignment (R)

Definition 7 (Reality Alignment): R is the expected log predictive likelihood:

$$R = \mathbb{E}[\log p(y \mid X)]$$

where $p(y \mid X)$ is the system's predictive distribution over outcomes y given state X. Higher R indicates better predictive accuracy. This is a standard measure of predictive performance; the label "reality alignment" is a philosophical interpretation.

Direction-dependence: The framework interprets R as potentially direction-dependent: $R_{A \to B} \neq R_{B \to A}$. This captures the asymmetry found in Berglund et al. (2024), where models trained on "A is B" fail to generalize to "B is A."

This interpretation is a framework-level claim.

Note on integration: Among the core variables, R is the least integrated with the trajectory-based formalism. Unlike κ, B, and $\tilde{B}$, which are directly derived from or related to DT, R is imported from Bayesian statistics. A more complete theoretical derivation of R from the same dynamical principles—perhaps as an information-theoretic functional of the occupation measure—remains an open direction for future work.

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5. Theoretical Framework

5.1 Relationship Between DT, P_{topo}, and $E(t)$

Functional	What It Measures	Regime
$DT(x)$	Cumulative deviation from attractor	All systems
$P_{topo}(t)$	Topological feature lifetime	Systems with topological structure
$E(t)$	Rate of topological change	Learning systems

Hypothesis: In learning systems, DT and P_{topo} are positively correlated early in learning and negatively correlated late in learning. Turner & Barak (2023) demonstrate that RNNs develop attractors sequentially during training, which may correspond to phases of topological simplification. This is a testable prediction.

5.2 Relationship Between κ, γ, and $E(t)$

Hypothesis: In a learning system, the topological evolution rate $E(t)$ is monotonically related to κ only if the system is not in persistent chaos: $\partial E/\partial \kappa > 0$ (with E and κ measured on appropriate scales) in convergent regimes. In persistent chaos, $E(t)$ is monotonically related to γ: $\partial E/\partial \gamma > 0$. Correlation analysis provides a statistical test of these monotonicity relationships.

5.3 Adaptive Landscape (Heuristic Note)

The adaptive landscape $V(X,t)$ evolves as: $\dot{V} = g(X,V) - \lambda V + \xi(t)$

For gradient systems with $\dot{X} = -\nabla_X V(X)$, and assuming the dynamics remain within the basin where higher-order nonlinearities are negligible, the cumulative deviation functional can be approximated as: $D_T(x) \approx \int_0^T \|\nabla_X V(\phi_\tau(x),\tau)\|\, d\tau$

This is a local heuristic. A full derivation and integration into the core formalism is left for future work.

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6. Testable Predictions

6.1 Core Prediction

Prediction: In a learning system, $E(t)$ is monotonically related to κ in convergent regimes: $\partial E/\partial \kappa > 0$ (with E and κ measured on appropriate scales), and $\partial E/\partial \gamma > 0$ in persistent chaos. Correlation analysis provides a statistical test of this monotonicity: $\mathrm{Corr}(E(t),\kappa)>0 \implies \lambda_{\max}<0$ $\mathrm{Corr}(E(t),\gamma)>0 \implies \lambda_{\max}\approx 0$

Falsification: If $E(t)$ correlates with κ in all regimes, or with γ in all regimes, the prediction is falsified.

6.2 Secondary Prediction

Prediction: In systems with high R, D_T and P_{topo} are negatively correlated late in learning; in systems with low R, they are uncorrelated or positively correlated.

Falsification: If D_T and P_{topo} are negatively correlated in both high-R and low-R systems, the prediction is falsified.

6.3 Boundary Condition and Global Falsifier

Conjecture: We conjecture that the framework applies to any system satisfying:

- A. Well-defined state space.
- B. Subject to perturbations.
- C. Exhibits at least one identifiable attractor.
- D. Dynamics are observable and measurable.

Global Falsifier: The unified ontology claim collapses if a system is found where D_T, κ, and topological persistence are mutually independent across all regimes, and where R cannot be expressed as a functional of the trajectory or occupation measure. If such a system exists, the framework’s claim to unify persistence, stability, and reality alignment would be falsified.

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7. Experimental Design

7.1 System Choice

Train a CNN on MNIST or CIFAR-10. Use latent activation manifolds for topological analysis.

Justification: Karuppiah, Nazreen Banu et al. (2026) demonstrate the use of persistent homology on activations to study feature learning and generalization. Turner & Barak (2023) show that RNNs develop attractors sequentially, providing a controlled setting for studying topological evolution during learning.

7.2 Variable Measurement

Variable	Protocol
$DT(x)$	Sample weights; compute distance to final attractor; integrate.
$P\text{topo}(t)$	Compute persistent homology on latent activations; sum feature lifetimes.
$E(t)$	Finite differences of $P\text{topo}(t)$.
κ	Perturb weights; measure recovery time τ; $\kappa=1/\tau$.
γ	Compute average drift rate during training.
R	Cross-domain generalization accuracy.

7.3 Statistical Analysis

- Correlate $E(t)$ with κ and γ conditional on regime.
- Pre-register thresholds and sample size.

Note on future empirical work: A full empirical validation would require pre-registration with specified sample size, significance thresholds, power analysis, and robustness checks. These are planned for subsequent work.

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8. Discussion

8.1 Implications

The paper provides a candidate formalization with defined variables, mathematical properties, and testable predictions. The mathematical properties of DT establish its relationship to κ and provide a foundation for the framework's core claims.

8.2 Limitations

- P_{topo} is computationally expensive.
- The framework is a meta-theory, not a complete domain-specific theory.
- Variables may be confounded; causal inference requires controlled experiments.
- The κ/γ regime distinction is proposed and requires empirical validation.

8.3 Future Work

- Empirical validation of predictions.
- Formal derivation of relationships from first principles.
- Extension to other domains.
- Computational efficiency improvements.

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9. Conclusion

This paper proposes a candidate formalization for the attractor framework. The central mathematical innovation is treating persistence as a functional defined over trajectories—$D_T(x)=\int_0^T d(\phi_\tau(x),\mathcal{A})\,d\tau$—rather than as a scalar property of states. We defined the cumulative deviation functional D_T, the topological persistence functional $P_{topo}(t)$, and the topological evolution rate $E(t)$. We proved several mathematical properties of D_T, including non-negativity, monotonicity, additivity, Lipschitz continuity, and a bound relating D_∞ to κ: $D_\infty(x)\le C_\kappa d(x,\mathcal{A})$. We established connections to dynamic programming and ergodic theory. We unified the variable set with operational definitions. We derived testable predictions and provided a falsifiable experimental protocol.

The framework now admits formal definitions, operational variables, and empirical tests. The next step is empirical validation.

Appendix A: Possible Extensions

from Larose (2025) — Unverified Source

Note: The following source has not been independently verified. It is included for completeness and as a potential direction for future exploration, but should not be treated as established.

Larose (2025) develops a framework for recursive deformation systems. Two constructs are potentially relevant:

Constraint Functional: $C(X) = \int_{\text{trajectory}} \|\nabla\Phi\| \, d\tau$, measuring cumulative irreversible deformation.

Persistence Invariant: $I_p = \oint R \, d\Phi$, a topological invariant.

These are not yet integrated into the core framework and are presented here for completeness and future exploration. They should be treated as unverified candidate extensions.

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