

The Persistence Functional: A Mathematical Measure of Attractor Resilience

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Abstract

The attractor framework says that **persistence under disturbance** is the basic mark of reality.

To turn this idea into a formal science, we introduce the **persistence functional** $P(x)P(x)$.

$P(x)P(x)$ is a single number that measures:

- How deep a state is inside an attractor basin.
- How quickly it returns after a knock.

We define $P(x)P(x)$ for three different kinds of systems:

1. **Deterministic dissipative systems** – here PP is linked to Lyapunov exponents and basin stability.
2. **Stochastic systems** – here PP is linked to escape time and quasipotential.
3. **Information-theoretic systems** – here PP is linked to negative free energy or mutual information.

The **recovery rate** $-P'/P - P'/P$ is a universal sign of **critical**

slowing down – a warning that a system is about to tip.

We also discuss limitations: resilience may depend on direction (“anisotropic”), and multiple timescales may need **vector** or **tensor** persistence. We list open mathematical problems.

This paper is a **roadmap**, not a finished theory.

1. Introduction

In the attractor framework, **persistence under disturbance** is central. But we have not had a single number to say *how persistent* a state is.

The **persistence functional** $P(x)$ aims to fill that gap.

What $P(x)$ should do:

- $P(x) > 0$ for states inside an attractor basin.
- For a **conservative attractor** (like a free electron), P is maximal (normalised to 1).
- For a **dissipative attractor**, P drops after a disturbance and then recovers.

The recovery rate $-P'/P$ equals:

- the negative of the largest Lyapunov exponent (for deterministic systems)
 - the inverse return time (for stochastic systems)
 - the rate of information loss (for informational systems)
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- P falls as the system approaches a **bifurcation**, giving early warning.

We do **not** give one universal formula. Instead, we give a **family** of definitions, each suited to a different type of

system, all united by the same purpose – measuring resilience.

2. Deterministic Dissipative Systems

Consider a smooth system $\dot{x}=f(x)$ with a stable attractor A and its basin $B(A)$.

A natural candidate for $P(x)$ uses a **Lyapunov function** $V(x)$ – a kind of energy that always decreases inside the basin ($\dot{V}<0$).

We define: $P(x) = 1 - \frac{V(x) - V_{\min}}{V_{\max} - V_{\min}}$

This gives $P=1$ on the attractor and $P \rightarrow 0$ at the basin boundary.

Near the attractor, the recovery rate is related to the **largest Lyapunov exponent** λ_1 : $-\dot{P}/P \approx -\lambda_1$

When the system approaches a tipping point, $\lambda_1 \rightarrow 0^-$, so the recovery rate slows down – this is **critical slowing down**.

Conclusion: For deterministic systems, P can be built from a Lyapunov function. The recovery rate equals the negative of the largest Lyapunov exponent.

3. Stochastic Systems

When noise is present, persistence is about how long it takes to escape from the basin.

The **mean first passage time** $\tau(x)$ – the average time to leave – is a natural measure.

We define: $P(x) = \tau(x) / \tau_{\max}$ $P(x) = \tau(x) / \tau_{\max}$

where $\tau_{\max}$ is the value at the attractor.

For weak noise, $\tau(x)$ grows exponentially with the **quasipotential** $U(x)$ (Freidlin–Wentzell theory): $\tau(x) \sim e^{U(x)/\epsilon}$

So: $P(x) \propto e^{-(U_{\max} - U(x))/\epsilon}$

The recovery rate is the inverse of the return time. As a tipping point is approached, the return time diverges, and the recovery rate goes to zero. This again gives **critical slowing down** – rising variance and autocorrelation.

Conclusion: For stochastic systems, P is proportional to the mean exit time (or the exponential of the quasipotential). This connects persistence to large deviation theory.

4. Information-Theoretic Systems

For systems where information matters (neural, cognitive, social), we can define persistence using **mutual information** between past and future.

Let $I_{\text{past}, \text{future}}$ be the **predictive information**. Then: $P(t) = I(\text{past}; \text{future at time } t)$ or $P = e^{-\text{surprisal}}$

The decay of $P(t)$ over time measures **memory loss**.

Landauer's principle connects information loss to entropy production: $P'/P \leq -S' / k_B \ln 2$ $P'/P \leq -k_B \ln 2 S'$

Alternatively, in the **free energy principle** (Friston), the negative free energy $-F$ acts like a Lyapunov function. We can set: $P = e^{-F/kT}$ or $P = -F$

Then $-P'/P - P'/P$ is the rate of free energy minimisation, which slows near bifurcations.

Conclusion: For information-theoretic systems, PP can be defined via mutual information decay or negative free energy, linking persistence to entropy production and predictive coding.

5. Unifying Recovery Rate and Critical Slowing Down

Across all types of systems, the **recovery rate** $\lambda_{\text{rec}} = -P'/P$ (just after a small disturbance) is a universal indicator:

- **Deterministic dissipative:** $\lambda_{\text{rec}} = -\lambda_1$ (absolute value of the largest Lyapunov exponent)
- **Stochastic:** λ_{rec} = inverse of the return time, related to the quasipotential's curvature
- **Information-theoretic:** λ_{rec} = rate of free energy minimisation or information loss

As the system approaches a bifurcation, $\lambda_{\text{rec}} \rightarrow 0$. This is **critical slowing down**.

It shows up as rising lag-1 autocorrelation and variance (Scheffer et al., 2009).

So PP and its recovery rate give early warnings.

6. Normalisation for Conservative

Attractors

For a perfect **conservative attractor** (e.g., an electron in its ground state, no decay), the persistence functional should be constant and maximal: $P_{\text{cons}} = 1$ for all times t .

No recovery rate is defined (or it is zero). This anchors the scale.

For **emergent approximate conservative systems** (like atomic clocks), PP is very close to 1 and decays extremely slowly.

7. Limitations – Scalar Collapse and Anisotropic Resilience

A single scalar $P(x)$ may not be enough for systems where resilience is **anisotropic** – that is, recovery speed depends on the direction of the perturbation.

High-dimensional systems can have **multiple timescales** (fast and slow modes). A scalar average can miss important structure.

Future work may need:

- **Vector persistence** – a list of recovery rates along different directions.
- **Tensor persistence** – a metric that captures the full shape of the basin.
- **Persistence manifold** – the geometry of the basin in state space.

We accept this limitation. The scalar PP is a useful first approximation for systems with isotropic resilience or for early-warning applications where a single number is enough.

For complex systems, a multidimensional generalisation is an open research problem.

8. Open Mathematical Problems

1. **Derive $P(x)$ from first principles** for a given class of systems (e.g., from a variational principle).
 2. **Prove that $-P'/P = \lambda_1$** for a wide class of dissipative systems.
 3. **Extend the definition to systems with multiple attractors and chaotic basins** (where basin stability is fractal).
 4. **Establish a rigorous relationship between PP and the mutual information decay rate** for non-equilibrium processes.
 5. **Formulate a universal persistence functional** that works across all regimes – or prove it's impossible.
 6. **Test the predictive power of PP** in controlled experiments (e.g., ecological microcosms, neural cultures, social media sentiment).
 7. **Develop vector/tensor persistence** for anisotropic resilience.
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9. Conclusion

The persistence functional $P(x)$ gives a mathematical language for attractor resilience.

We have given **operational definitions** for three regimes:

- **Deterministic dissipative** → Lyapunov / basin stability

- **Stochastic** → escape time / quasipotential
- **Information-theoretic** → mutual information / free energy

The **recovery rate** $-P'/P - P'/P$ unifies critical slowing down across all these domains.

We have explicitly noted **limitations** (scalar collapse, anisotropy) as open problems.

This paper is a **roadmap**, not a final theory. The framework now has a quantitative step.

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