[illegible]

[illegible] — 2026[illegible]6[illegible]

Robert Galida
[illegible]

[illegible]

[illegible]——κ[illegible] B[illegible] R[illegible]——[illegible]

[illegible]——$D_T(x)=\int_0^T d(\phi_\tau(x),A)\,d\tau$——[illegible] D_T[illegible] T [illegible] D_∞[illegible] κ [illegible] $D_\infty(x)\leq \frac{C}{\kappa} d(x,A)$[illegible] $P_{\text{topo}}(t)$[illegible] $E(t)$[illegible]

[illegible] κ [illegible] $1/\tau$[illegible] γ [illegible] B [illegible] $\tilde{B}$ [illegible] R [illegible] $E(t)$ [illegible] κ [illegible] γ [illegible] falsifiable [illegible]

[illegible]

[illegible]

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1. [illegible]

[illegible]ECM[illegible]HRV[illegible] κ[illegible] B[illegible] R[illegible]

[illegible] $D_T(x)=\int_0^T d(\phi_\tau(x),A)\,d\tau$ [illegible] $\int L(q,\dot{q})\,dt$ [illegible]

[illegible] $D_T(x)$ [illegible] κ [illegible] $P_{topo}(t)$ [illegible] $E(t)$ [illegible]

1.1 [illegible]

[illegible]——[illegible]/[illegible]

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2. [illegible]

[illegible] X [illegible] $\|\cdot\|$ [illegible] $\phi_\tau(x)$ [illegible] $x\in X$ [illegible] $\tau=0$ [illegible] $A\subseteq X$ [illegible] $d(\phi_\tau(x),A)$ [illegible] ϕ_τ [illegible] $\phi_{t+s}=\phi_t\circ\phi_s$ [illegible] $t,s\geq 0$ [illegible] $\phi_0=\mathrm{id}$ [illegible] $d(\phi_\tau(x),A)\in L^1([0,T])$ [illegible] T [illegible] D_T [illegible]

[illegible] $d(x,A)=\inf_{a\in A}\|x-a\|$ [illegible]

[illegible]

2.1 [illegible]

[illegible] **1** [illegible] $T>0$ [illegible] $D_T(x)=\int_0^T d(\phi_\tau(x),A)\,d\tau$

[illegible] $D_T(x)$ [illegible] $[0,T]$ [illegible] $\int\|\dot{\phi}_\tau(x)\|\,d\tau$ [illegible]

[illegible]

□□□□□ D_T□ □□□□□□□□□□□D_∞□ □□□□□□□□□□□□□□

□□□ D_T□ □□□□□□□□□□□□□□□□□□□□□□□□□□□□D_T□ □□□□□□□□□□□□□□□□□□□ □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□ □□□□□□□ T □□□□□□□$\mu_T(B)=\int_0^T \mathbf{1}_B(\phi_\tau(x))\,d\tau$

□□□ $B\subseteq X$□□□$D_T(x)=\int_X d(y,A)\,d\mu_T(y)$

□□ D_T□ □□□ □□□□□□ Ruelle (1989) □ Bowen (1975)□

2.1.1 □□□□□ L^1 □□□□□

□□ L^1 □□□□□□□□□□□□□□□□□

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□□□□□□□□□□□□□□□ d_p□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

2.2 □□□□□□□

□ $X_\tau=\{\phi_s(x):s\in[0,\tau]\}$ □□□□ τ □□□□□ □ $PH_k(X_\tau)$ □□□ X_τ□ □□□ ϵ □ k □□□□□□□□□□□□□□□□□□□□□□□□ □□□□□□ b □□□□□□□ d□□□□□ $d-b$□□□□□□□□□□□□□□□□ Edelsbrunner & Harer (2010) □ Carlsson (2009)□

□□ **2**□□□□□□□□□□□ □□□□□□□□□□□□□□□□□□ □ $t\geq 0$□$P_{topo}(t)=\int_0^t \sum_{k\geq 0}\sum_{(b,d)\in PH_k(X_\tau)}(d-b)\,d\tau$

k≥0∑□(b,d)∈PHk□(Xτ□)∑□(d−b)dτ

□□ τ↦PHk(Xτ)τ↦PHk□(Xτ□) □□□ Carlsson & Zomorodian, 2009□□

□□□ Ptopo(t)Ptopo□(t) □□□□ tt □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□ DTDT□ □□□ DTDT□ □□□□□□□□□□□□□□□□□□□□□□□□□□□□——□□□□□□□□□□□□□□□□□□□□——□□□□□□□□□□□□□□□□□□□□□□□□□□□

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2.3 □□□□□

□□ 3□□□□□□□□□ □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□ □□E(t)=ddtPtopo(t)E(t)=dtd□Ptopo□(t)

□□□□□□□□□□□ E(t)≈ΔPtopoΔtE(t)≈ΔtΔPtopo□□ □□□□□□□□

□□□ E(t)E(t) □□□□□□□□□□□□□□□□□□□□□□□□□□□□ E(t)E(t) □□□□□□□□□□□□□□□ □ E(t)E(t) □□□□□□□□□□□□□□□□E(t)≈0E(t)≈0 □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□ □□□□□□□□□□□□□□□□□□

□□□□□ Karuppiah, Nazreen Banu □□ (2026) □□□□□□□□□□□□□□□□□□□□□Turner & Barak (2023) □□ RNN □□□ □□ E(t)E(t) □□□□□□□□□□□□□□□□□□□□□□□□□□□□□

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3. □□□□□□□□□□□□□□□□□

□□□□ DTDT□ □□□□□□□□□□□□□□□□□□□□□□□□□□

3.1 □□□

□□ **1**□□□□□□ □□□ $x\in X$ □□□ $T\ge 0$□$D_T(x)\ge 0$

□□□□□□□□ $\phi_\tau(x)\in A$ □□□□□ $\tau\in[0,T]$ □□□

□□□ □□□□□□□□□ $d(\phi_\tau(x),A)$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

3.2 □□ T □□□□

□□ **2**□□□□□□ □□ x□$D_T(x)$ □□ T □□□□□□$D_{T_2}(x)\ge D_{T_1}(x)$ □ $T_2\ge T_1$□

□□□ □ $T_2\ge T_1$ □$D_{T_2}(x)=\int_0^{T_1} d(\phi_\tau(x),A)\,d\tau+\int_{T_1}^{T_2} d(\phi_\tau(x),A)\,d\tau$

□□□□□□□□□□□ 1□□□□ $D_{T_2}(x)\ge D_{T_1}(x)$□

□□□ □□□□□□□ $\tau_0<T$ □□□□□□□□□□□$D_T(x)=D_{\tau_0}(x)$□□□ $T\ge\tau_0$□

3.3 □□□

□□ **3**□□□□□□ □□□ $T,S\ge 0$□$D_{T+S}(x)=D_T(x)+D_S(\phi_T(x))$

□

□$D_{T+S}(x)=\int_0^{T+S} d(\phi_\tau(x),A)\,d\tau=\int_0^{T} d(\phi_\tau(x),A)\,d\tau+\int_T^{T+S} d(\phi_\tau(x),A)\,d\tau=D_T(x)+\int_0^{S} d(\phi_{\tau+T}(x),A)\,d\tau=D_T(x)+\int_0^{S} d(\phi_\tau(\phi_T(x)),A)\,d\tau$□□□□□□$=D_T(x)+D_S(\phi_T(x))$□

□□□□□ D_T□ □□□□□□□□□□□□□□□□□□□□□□□□□□

3.4 □□□□□□□□□□

□□□□□ DT+S(x)=DT(x)+DS(ϕT(x))*DT*+*S*□(*x*)=*DT*□(*x*)+*DS*□(*ϕT*□(*x*)) □□□□□□□□□□□□□□□□□□□□□ X˙=f(X,u)*X*˙=*f*(*X*,*u*) □□□ u∈U*u*∈U□□□□ V(x)=inf□uD∞(x)*V*(*x*)=inf*u*□*D*∞□(*x*) □□□□□ Hamilton-Jacobi-Bellman □□□0=inf□u{d(x,A)+∇V(x)·f(x,u)}0=*u*inf{*d*(*x*,A)+∇*V*(*x*)·*f*(*x*,*u*)}

□□□□□□□□□□□□□□□□□□□□ DT*DT*□ □□□□□□□□□□□□□□□□□□□□□□□□□□□□

3.5 □□□□□□□□□□□□□□

□□ **4□DT*DT*□** □□□□□□□□□□□ □□□ ϕτ*ϕτ*□ □ x*x* □□□□□□□□□□□□ L*L*□□ □ϕτ(x)−ϕτ(y)□≤eLτ□x−y□□*ϕτ*□(*x*)−*ϕτ*□(*y*)□≤*eLτ*□*x*−*y*□□□□□□□ AA □□□□□ x,y*x*,*y*□□DT(x)−DT(y)□≤∫0TeLτ dτ □x−y□=eLT−1L□x−y□□*DT*□(*x*)−*DT*(*y*)□≤∫0*T*□*eLτdτ*□*x*−*y*□=*LeLT*−1□□*x*−*y*□

□□□ □□□□□□□□□ d(·,A)*d*(·,A) □ 1-□□□□□□□□□ x,y∈X*x*,*y*∈X,□d(x,A)−d(y,A)□≤□x−y□□*d*(*x*,A)−*d*(*y*,A)□≤□*x*−*y*□

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□□□DT(x)−DT(y)□≤∫0T□d(ϕτ(x),A)−d(ϕτ(y),A)□dτ≤∫0T□ϕτ(x)−ϕτ(y)□dτ≤∫0TeLτ□x−y□dτ=eLT−1L□x−y□□*DT*(*x*)−*DT*□(*y*)□□≤∫0*T*□□*d*(*ϕτ*□(*x*),A)−*d*(*ϕτ*□(*y*),A)□*dτ*≤∫0*T*□□*ϕτ*□(*x*)−*ϕτ*(*y*)□*dτ*≤∫0*T*□*eLτ*□*x*−*y*□*dτ*=*LeLT*−1□□*x*−*y*□□

□□□ □□□□□ DT*DT*□ □□□□□□□□□□□□□□□□□□□□□□□ DT*DT*□ □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

3.6 □□□□□

□□ **1**□□□□□□□□ □□□□□□ d(ϕτ(x),A)*d*(*ϕτ*□(*x*),A) □□ τ*τ* □□□□□ddTDT(x)=d(ϕT(x),A)*dTd*□*DT*□(*x*)=*d*(*ϕT*□(*x*),A)

3.7 □□□□

□□ **5**□□□□□□□ □□□□□□□□□ $\nu_T=\mu_T/T$ □ $T\to\infty$ □□□□□□□□□□□ μ□□□$\lim_{T\to\infty}\frac{1}{T}D_T(x)=\int_X d(y,A)\,d\mu(y)$

□□□ □□□□□□□□ $D_T(x)=\int d(y,A)\,d\mu_T(y)=T\int d(y,A)\,d\nu_T(y)$□$\nu_T$□ □ μ □□□□ □ $d(\cdot,A)$ □□□□/□□□□□□□□□□

□□□□□□ $d(\cdot,A)$ □□□□□□□□□□□□□□□□□□□□□□□□ Bowen (1975) □ Ruelle (1989)□

3.8 □□□□□□□□

□□ **2**□□□□□□□□□□□□ □□□ $\phi_\tau(x)$ □□□□□ $\kappa>0$ □□□□□ □ A□$d(\phi_\tau(x),A)\le Ce^{-\kappa\tau}d(x,A)$

□□□□□ $C<\infty$□□□□ $\tau\ge0$□□□$D_\infty(x)=\int_0^\infty d(\phi_\tau(x),A)\,d\tau\le\frac{C}{\kappa}d(x,A)$

□□□$D_\infty(x)=\int_0^\infty d(\phi_\tau(x),A)\,d\tau\le\int_0^\infty Ce^{-\kappa\tau}d(x,A)\,d\tau=Cd(x,A)\int_0^\infty e^{-\kappa\tau}\,d\tau=\frac{C}{\kappa}d(x,A)$

□□□ □□□□□□□ κ □□□□□□□□$D_\infty(x)\le\frac{1}{\kappa}d(x,A)$□□ $C=1$ □□□

□□□ □□□□□□□ $D_\infty<\infty$□□□□□□□□□□□□□□□□□□□□□□ D_∞□□

3.9 □□□□

□□ **1**□□□□□□□ □□□□□□□□ C □□□□□□□□□□□□□□□□ κ □□□$\kappa \le \frac{D_\infty(x)}{C\, d(x,A)}$□

□□ $C=1$ □□□□□□□□□□□□□□□□/□□□□□□□□□□□□□$\kappa \le \frac{D_\infty(x)}{d(x,A)}$□

□□□ □□□ 2□$D_\infty(x) \le \kappa C\, d(x,A)$□□□□□ $\kappa \le \frac{D_\infty(x)}{C\, d(x,A)}$□□□ $C=1$ □□□□ $\kappa \le \frac{D_\infty(x)}{d(x,A)}$□□

□□□ □□□□□□□□□□□□□□ κ□□□□□□□□□□□□□□□□ κ□□□□□□□ D_T□ □ κ □□□□□□□□□□ C □□□□□□□□□□□□□□□□□□

3.10 □□□□□□

□□ **6**□□□□□□□ □□□ $\epsilon > 0$□□□□□ T_ϵ□□□□□□□ $T > T_\epsilon$ □□$|D_T(x) - D_\infty(x)| \le \epsilon$

□□□ □□□□□□□ 2 □□□□□□□□□□□□□□□□□□□□□□□□□□□ $\int_T^\infty d(\phi_\tau(x), A)\, d\tau$ □□□□□□□□□□□ T □□□□

3.11 □□□□

□□	□□
□□□	$D_T(x) \ge 0$
□□□	$D_{T_2}(x) \ge D_{T_1}(x)$ □ $T_2 \ge T_1$□
□□□	$D_{T+S}(x) = D_T(x) + D_S(\phi_T(x))$

□□	□□		
□□□□□□□	(	D_T(x) − D_T(y)	\leq \frac{e^{LT} − 1}{L} |x − y|)
□□□□□	ddTDT(x)=d(ϕT(x),A)*dTd*□*DT*□(*x*)=*d*(*ϕT*(*x*),A)		
□□□□	lim□T→∞1TDT(x)=∫d(y,A) dμ(y)lim*T*→∞*T*1□*DT*□(*x*)=∫*d*(*y*,A)*dμ*(*y*)		
□□□□□□□□D∞□	D∞(x)≤Cκd(x,A)*D*∞□(*x*)≤*κC*□*d*(*x*,A)		
□□□□□□□	κ≤C d(x,A)D∞(x)*κ*≤*D*∞□(*x*)*Cd*(*x*,A)□		
□□□□C=1□	κ≤d(x,A)D∞(x)*κ*≤*D*∞□(*x*)*d*(*x*,A)□		
□□□□□□	DT(x)→D∞(x)*DT*□(*x*)→*D*∞□(*x*) □ T→∞*T*→∞		

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4. □□□□□

□□□□□□□□□□□□□□□□□□□□□□□□

4.1 □□□□□ (κ*κ*)

□□ 4□□□□□□□□ κ*κ* □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□ κ=1/τ*κ*=1/*τ*□□□ τ*τ* □□□

□ **DT**□ □□□□ □□□ 1□□□□□□□ $d(x,A)$ □□□□ $\kappa \le C\, D_\infty(x)\, d(x,A)$□□

□□ **κ** □□□□□□ □□□□□κ □□□□□□□□□□□□□□□□□□□□□□□ DT□ □□□□□□□ κ□□□□□□□□□□□□□□□

4.2 □□□ (γ) — □□□□□□□

□□ **5**□□□□□□ □□□□□□□□□□□□□□ λ_{max} □□□□□□□□□□□□□□□□

□□	**λ_{max}**	**κ**	**γ**	□□
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□□□□	>0	□□□	>0	□□

□□□ $\lambda_{max} < -0.01$□□$\lvert\lambda_{max}\rvert \le 0.01$□□ $\lambda_{max} > 0.01$□□□□□□ 1/epoch □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

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□□□ □□ κ □ γ □□□□□□□ κ □□□□□□□□ γ□□□□□□□□□

4.3 □□□□ (B) □□□□□ ($\tilde{B}$)

□□ **6a**□□□□□ — □□□□ B □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$B = V(□□) - V(□□□)$

□□□□□□□□□□□□□□□□

□□ **6b**□□□□□□□ □□□□□□□□□□□□□$\tilde{B} = \min_{x \in \partial B} DT(x)$

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4.4 □□□□ (R)

□□ 7□□□□□□□ R □□□□□□□□□□□$R=\mathbb{E}[\log p(y∣X)]$

□□ $p(y∣X)$ □□□□□□□□□ X □□□ y □□□□□□□□□□ R □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□”□□□□”□□□□□□□□

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5.1 DT□□P_{topo}□ □ $E(t)$ □□□□□

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$DT(x)$	□□□□□□□□□□	□□□□
$P_{topo}(t)$	□□□□□□	□□□□□□□□□□
$E(t)$	□□□□□	□□□□

□□□：□□□□□□□DT□ □ P_{topo}□ □□□□□□□□□□□□□□□□□□□□Turner & Barak (2023) □□□ RNN □□

5.2 κ□γ □ $E(t)$ □□□□□

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5.3 □□□□□□□□□□□□

□□□□□ $V(X,t)$ □□□□$\dot{V}=g(X,V)-\lambda V+\xi(t)$

□□□□□□ $\dot{X}=-\nabla_X V(X)$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□ □□$D_T(x)\approx\int_0^T$□$\nabla_X V(\phi_\tau(x),\tau)$□ $d\tau$

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6. □□□□

6.1 □□□□

□□□ □□□□□□$E(t)$ □□□□□□ κ □□□□□$\partial E/\partial \kappa>0$□$E$ □ κ □□□ □□□□□□□□□□□□□□□ $\partial E/\partial \gamma>0$□□□□□□□□□□□□□□□□□□□□□ □□$\mathrm{Corr}(E(t),\kappa)>0$ □ $\lambda_{\max}$□<0 $\mathrm{Corr}(E(t),\gamma)>0$ □ $\lambda_{\max}$□≈ 0

□□□ □□ $E(t)$ □□□□□□ κ □□□□□□□□□□ γ □□□□□□□□

6.2 □□□□

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□□□ □□ D_T□ □ P_{topo}□ □□ R □□ R □□□□□□□□□□□□□

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7. □□□□

7.1 □□□□

□ MNIST □ CIFAR-10 □□□ CNN□□□□□□□□□□□□□□□□

□□□ Karuppiah, Nazreen Banu □□ (2026) □□□□□□□□□□□□□□□□□□□□□□□□□Turner & Barak (2023) □□ RNN □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

7.2 □□□□

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$DT(x)$	□□□□□□□□□□□□□□□□□□□□
$P_{topo}(t)$	□□□□□□□□□□□□□□□□□□□□□
$E(t)$	$P_{topo}(t)$ □□□□□□
κ	□□□□□□□□□□□ τ□$\kappa=1/\tau$□
γ	□□□□□□□□□□□□□

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R	□□□□□□□□

7.3 □□□□

- □□□□□□□□ $E(t)$ □ κ □ γ□
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8. □□

8.1 □□

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8.2 □□□

- P_{topo}□ □□□□□□
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□□□□ $C(X)=\int$□□□$\nabla\Phi$□ $d\tau$□□□□□□□□□

□□□□□ I_p=□$R\,d\Phi$□□□□□□□

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